This chapter provides a framework for addressing situations where expectations in *uncertain* situations are important elements. Our previous models have not taken seriously the problem of uncertainty. Where agent's expectations about future variables were involved and these expectations were assumed to be model-consistent ("rational"), we only considered a special case: perfect foresight. Shocks were treated in a peculiar, almost self-contradictory way: they might occur, but only as a complete surprise, a one-off event. Agents' expectations and actions never incorporated that new shocks could arrive.

We will now allow recurrent shocks to take place. The environment in which the economic agents act will be considered inherently uncertain. How can this be modeled and how can we solve the resultant models? Since it is easier to model uncertainty in discrete rather than continuous time, we examine uncertainty and expectations in a discrete time framework.

Our main emphasis will be on the hypothesis that when facing uncertainty a predominant fraction of the economic agents form "rational expectations" in the sense of making probabilistic forecasts which coincide with the forecast calculated on the basis of the "relevant economic model". As an example of application of this hypothesis we shall consider the issue of neutrality versus non-neutrality of money in a framework where all agents have rational expectations whereas nominal wages are not perfectly flexible.

But first a summary of simple traditional expectation formation formulas.

25.1 Preliminaries

Here we consider some "mechanistic" expectation formation formulas that have been used to describe day-to-day expectations of people who do not think much about the statistical properties of their economic environment.

25.1.1 Simple expectation formation hypotheses

One simple supposition is that expectations change gradually to correct past expectation errors. Let P_t denote the general price level in period t and $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ the corresponding inflation rate. Further, let $\pi^e_{t-1,t}$ denote the "subjective expectation", formed in period $t-1$, of π_t, i.e., the inflation rate from period $t-1$ to period t. We may think of the "subjective expectation" as the expected value in a vaguely defined subjective conditional probability distribution.

The hypothesis of *adaptive expectations* (the AE hypothesis) says that the expectation is revised in proportion to the past expectation error,

$$\pi^e_{t-1,t} = \pi^e_{t-2,t-1} + \lambda(\pi_{t-1} - \pi^e_{t-2,t-1}), \quad 0 < \lambda \leq 1, \tag{25.1}$$

where the parameter λ is called the adjustment speed. If $\lambda = 1$, the formula reduces to

$$\pi^e_{t-1,t} = \pi_{t-1}. \tag{25.2}$$

This limiting case is known as *static expectations* or *myopic expectations:* the subjective expectation is that the inflation rate will remain the same. As we shall see, *if* inflation follows a random walk, this subjective expectation is in fact the "rational expectation".

We may write (25.1) on the alternative form

$$\pi^e_{t-1,t} = \lambda\pi_{t-1} + (1-\lambda)\pi^e_{t-2,t-1}. \tag{25.3}$$

This says that the expected value concerning this period (period t) is a weighted average of the actual value for the last period and the expected value for the last period. By backward substitution we find

$$\begin{aligned}
\pi^e_{t-1,t} &= \lambda\pi_{t-1} + (1-\lambda)[\lambda\pi_{t-2} + (1-\lambda)\pi^e_{t-3,t-2}] \\
&= \lambda\pi_{t-1} + (1-\lambda)\lambda\pi_{t-2} + (1-\lambda)^2[\lambda\pi_{t-3} + (1-\lambda)\pi^e_{t-4,t-3}] \\
&= \lambda\sum_{i=1}^{n}(1-\lambda)^{i-1}\pi_{t-i} + (1-\lambda)^n\pi^e_{t-n-1,t-n},
\end{aligned}$$

assuming the adjustment speed has been the same in the previous n periods.

Since $(1-\lambda)^n \to 0$ for $n \to \infty$, we have (for $\pi^e_{t-n-1,t-n}$ bounded as $n \to \infty$),

$$\pi^e_{t-1,t} = \lambda\sum_{i=1}^{\infty}(1-\lambda)^{i-1}\pi_{t-i}. \tag{25.4}$$

Thus, according to the AE hypothesis with $0 < \lambda < 1$, the expected inflation rate is a weighted average of the historical inflation rates back in time. The weights are geometrically declining with increasing time distance from the current period. The weights sum to one: $\sum_{i=1}^{\infty}\lambda(1-\lambda)^{i-1} = \lambda(1-(1-\lambda))^{-1} = 1$.

The formula (25.4) can be generalized to the *general backward-looking expectations* formula,

$$\pi^e_{t-1,t} = \sum_{i=1}^{\infty} w_i\pi_{t-i}, \qquad \text{where } \sum_{i=1}^{\infty} w_i = 1. \tag{25.5}$$

If the weights w_i in (25.5) satisfy $w_i = \lambda(1-\lambda)^{i-1}$, $i = 1, 2, \ldots$, we get the AE formula (25.4). If the weights are

$$w_1 = 1+\beta,\ w_2 = -\beta,\ w_i = 0 \text{ for } i = 3, 4, \ldots,$$

we get

$$\pi^e_{t-1,t} = (1+\beta)\pi_{t-1} - \beta\pi_{t-2} = \pi_{t-1} + \beta(\pi_{t-1} - \pi_{t-2}). \tag{25.6}$$

This is called the hypothesis of *extrapolative expectations* and says:

if $\beta > 0$, then the recent direction of change in π is expected to continue;
if $\beta < 0$, then the recent direction of change in π is expected to be reversed;
if $\beta = 0$, then expectations are static as in (25.2).

As hinted, there *are* cases where for instance myopic expectations *are* "rational" (in a sense to be defined below). Example 2 below provides an example. But in many cases purely backward-looking formulas are too rigid, too mechanistic. They will often lead to systematic expectation errors to one side or the other. It seems implausible that humans should not then respond to this experience and revise their expectations formula. When expectations are about things that really matter for people, they are likely to listen to professional forecasters who build their forecasting on economic and statistical *models*. Such models are based on a formal probabilistic framework, take the interaction between different variables into account, and incorporate new information about future likely events.

Agents' expectations, whatever their nature, can enter a macroeconomic model in different ways. The next sub-section considers two basic alternatives.

25.1.2 Two model types

We first recapitulate a few concepts from statistics. A sequence $\{X_t;\ t = 0, 1, 2, \dots\}$ of random variables indexed by time is called a *stochastic process.* Often the index set $T = \{0, 1, 2, \dots\}$ is understood and we just write $\{X_t\}$ (or, if there is no risk of confusion, just X_t). A stochastic process $\{X_t\}$ is called *white noise* if for all t, X_t has zero expected value, constant variance, and zero covariance across time.[1] A stochastic process $\{X_t\}$ is called a *first-order autoregressive process,* abbreviated AR(1), if $X_t = \beta_0 + \beta_1 X_{t-1} + \varepsilon_t$, where β_0 and β_1 are constants, and $\{\varepsilon_t\}$ is white noise. If $|\beta_1| < 1$, then $\{X_t\}$ is called a *stationary* AR(1) process.

A stochastic process $\{X_t\}$ is called a *random walk* if $X_t = X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\}$ is white noise. Since here the implicit β_1 violates the requirement $|\beta_1| < 1$, the process is said to be *non-stationary.* But the first-difference, $\Delta X_t \equiv X_t - X_{t-1}$ is *stationary.*

[1] The expression "white noise" derives from electrotechnics. In electrotechnical systems signals will often be subject to noise. If this noise is arbitrary and has no dominating frequence, it looks like white light. The various colours correspond to a certain wave length, but white light is light which has all frequences (no dominating frequence).

Model type A: models with past expectations of current endogenous variables

Suppose a given macroeconomic model can be reduced to two key equations for each t, the first being

$$Y_t = a\ Y^e_{t-1,t} + c\ X_t, \qquad t = 1, 2, ..., \tag{25.7}$$

where Y_t is some endogenous variable (not necessarily GDP), $Y^e_{t-1,t}$ is the subjective expectation formed in period $t-1$, of the value of the variable Y in period t, a and c are given constant coefficients, and X_t is an exogenous random variable which follows some specified stochastic process.

The economic agents are in simple models assumed to hold the same expectations. Or, at least there is a dominating expectation, $Y^e_{t-1,t}$, in the market, sometimes called the "market expectation". What the equation (25.7) claims is that the endogenous variable, Y_t, depends, in a specified linear way, on the "generally held" expectation of Y_t, formed in the previous period. It is convenient to think of the outcome Y_t as being the aggregate result of agents' decisions and interaction in the market, the decisions being made at discrete points in time $\ldots$, $t-2$, $t-1$, $t, \ldots$, immediately after the uncertainty concerning the period in question has been resolved.

The second key equation specifies how the subjective expectation is formed. As an example, let us assume that the subjective expectation is myopic, i.e.,

$$Y^e_{t-1,t} = Y_{t-1}, \tag{25.8}$$

as in (25.2) above.

Then a *solution* to the model is a stochastic process for Y_t such that (25.7) holds, given the expectation formation (25.8) and the stochastic process which X_t follows. Substituting (25.8) into (25.7), we get

$$Y_t = aY_{t-1} + cX_t, \qquad t = 1, 2, \ldots. \tag{25.9}$$

If for instance $X_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and $\{\varepsilon_t\}$ is white noise, then the solution expressed in terms of the lagged Y is $Y_t = aY_{t-1} + c\bar{x} + c\varepsilon_t$. In Example 1 below a solution appears as a specification of the complete time path of Y_t, given Y_0.

EXAMPLE 1 *(imported raw materials and domestic price level)* Let the endogenous variable in (25.7) represent the domestic price level (the consumer price index) P_t, and let X_t be the price level of imported raw materials. Suppose the price level is determined through a markup on unit costs,

$$P_t = (1+\mu)(\lambda W_t + \eta X_t), \qquad 0 < \lambda < \frac{1}{1+\mu}, \tag{*}$$

where W_t is the nominal wage level in period $t = 1, 2, \ldots$, and λ and η are positive technical coefficients representing the assumed constant labor and raw materials requirements, respectively, per unit of output; μ is a constant markup. The upper inequality in (*) is imposed to avoid an exploding wage-price spiral. Assume further that workers in period $t-1$ negotiate next period's wage level, W_t, so as to achieve, in expected value, a certain target real wage which we, by proper choice of unit measurement for labor, normalize to 1, i.e.,

$$\frac{W_t}{P^e_{t-1,t}} = 1. \tag{**}$$

Substituting into (*), we get

$$P_t = a\ P^e_{t-1,t} + c\ X_t, \qquad 0 < a \equiv \lambda(1+\mu) < 1, 0 < c \equiv \eta(1+\mu), \quad t = 1, 2, \ldots.$$

Assuming myopic expectations, $P^e_{t-1,t} = P_{t-1}$, this gives the reduced-form equation $P_t = a\ P_{t-1} + cX_t$, for $t = 1, 2, \ldots$. By repeated application of this, starting with a given P_0, we find $P_t = a^t P_0 + c\sum_{i=1}^{t} a^{t-i} X_i$. Suppose $X_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a positive constant and $\{\varepsilon_t\}$ is white noise. Then the stochastic process followed by P_t is a stationary AR(1) process. We can express the solution to the model as a time path of P, given P_0 and the realized values of the noise term ε :

$$\begin{aligned} P_t &= a^t P_0 + c\sum_{i=1}^{t} a^{t-i}\bar{x} + c\sum_{i=1}^{t} a^{t-i}\varepsilon_i \\ &= (P_0 - P^*)\, a^t + P^* + c\sum_{i=1}^{t} a^{t-i}\varepsilon_i, \end{aligned} \tag{***}$$

where $P^* \equiv \frac{c\bar{x}}{1-a}$ and $t = 1, 2, \ldots$. In this derivation we have applied the rule for the first t terms of a geometric series.

Without shocks, since $0 < a < 1$, the price level converges to P^* for $t \to \infty$. Shocks to the price of imported raw materials result in transitory deviations from P^*, the persistence of which can be measured by a. The parameter a also governs the persistence of the systematic expectation error generated by myopic expectations in this model. Suppose that for a long time no shocks have occurred and P_t has settled down at P^*. Then, at $t = t_0$ a positive shock ε_{t_0} occurs so that $P_{t_0} = P^* + c\varepsilon_{t_0}$. If for $t = t_0 + 1$, $t_0 + 2$, ..., no new shocks occur, then, for $t \geq t_0$, $P_t = (P_{t_0} - P^*)\, a^t + P^*$ according to the rule in (***). And after t_0 the subjective expectation systematically exceeds the realization of P_t by the amount $(P_{t_0} - P^*)a^{(t-1-t_0)} - (P_{t_0} - P^*)a^{(t-t_0)} = (P_{t_0} - P^*)(1-a)a^{(t-1-t_0)}$. This systematic expectation bias arises because the expectation formation is mechanistic and does not consider how the system as a whole actually functions. □

Equation (25.7) can also be interpreted as a vector equation (such that Y_t and $Y^e_{t-1,t}$ are n-vectors, a is an $n \times n$ matrix, c an $n \times m$ matrix, and X an m-vector). The crucial feature is that the endogenous variables dated t *only* depend on previous expectations of date-t values of these variables and on the exogenous variables.

Models with past expectations of current endogenous variables will serve as our point of reference when considering the role of rational expectations below.

Model type B: models with forward-looking expectations

Another way in which agents' expectations may enter is exemplified by

$$Y_t = a\ Y^e_{t,t+1} + c\ X_t, \qquad a \neq 1, \qquad t = 0, 1, 2, ... \tag{25.10}$$

Here $Y^e_{t,t+1}$ is the subjective expectation, formed in period t, of the value of Y in period $t+1$. Example: the equity price today depends on current circumstances as well as what the equity price is expected to be tomorrow. Or more generally: the current expectation of a future value of an endogenous variable influences the current value of this variable. We name this the case of *forward-looking expectations*. (In "everyday language" also $Y^e_{t-1,t}$ in model type 1 can be said to be a forward-looking variable as seen from period $t-1$. But the dividing line between the two model types, (25.7) and (25.10), is whether *current* expectations of future values of the endogenous variables do or do not influence the current values of these.)

A complete model with forward-looking expectations will include an additional equation, specifying how the subjective expectation, $Y^e_{t,t+1}$, is formed. We might again impose the myopic expectations hypothesis, now taking this form:

$$Y^e_{t,t+1} = Y_t. \tag{25.11}$$

A *solution* to the model is a stochastic process for Y_t satisfying (25.10), given the stochastic process followed by X_t and given the specified expectation formation (25.11) and perhaps some additional restrictions in the form of boundary conditions. In the present case, where expectations are myopic, the solution simply is

$$Y_t = aY_t + cX_t = \frac{cX_t}{1-a} \qquad t = 0, 1, 2, ... \tag{25.12}$$

The case of forward-looking expectations is important in connection with many topics in macroeconomics, including aggregate fixed capital investment, evolution of asset prices, issues of asset price bubbles, etc.

In passing we note that in type A as well as type B models, it is the mean (in the subjective probability distribution) of the random variable(s) that enters.

This is typical of simple macroeconomic models which often do not require other measures such as the median, the mode, or higher-order moments. The latter, say the variance of X_t, may be included in advanced models where for instance behavior towards risk is important.

25.1.3 The model-consistent expectation

The concepts of a *rational expectation* and a *model-consistent expectation* are closely related, but not the same. We need the latter concept to be able to define the former.

Consider a stochastic model of type A, as represented by (25.7) combined with some given expectation formation, (25.8) say. Then the *model-consistent expectation* of the endogenous variable Y_t as seen from period $t-1$ is the mathematical *conditional expectation* that can be calculated on the basis of the model and available relevant data revealed up to and including period $t-1$. A common notation for this expectation is

$$E(Y_t|I_{t-1}), \tag{25.13}$$

where E is the expectation operator and I_{t-1} is the *information set* available at time $t-1$. We think of period $t-1$ as the half-open time interval $[t-1, t)$ and imagine that the uncertainty concerning the exogenous random variable X_{t-1} is resolved at time $t-1$. Unless otherwise indicated, we think of I_{t-1} as including knowledge of the realization of X_{t-1} and Y_{t-1}.

Letting Y_t be a continuous stochastic variable with range $(-\infty, \infty)$, the model-consistent expectation as seen from period $t-1$ is

$$E(Y_t|I_{t-1}) = \int_{-\infty}^{\infty} y_t f(y_t|I_{t-1}) dy_t, \tag{25.14}$$

where $f(y_t|I_{t-1})$ is the conditional probability density function for Y_t, given the model and the information set I_{t-1}.

The information set I_{t-1} may comprise knowledge of the realized values of X and Y since period 0 and up until (and including) period $t-1$. Instead of the abstract form (25.13) we can then write

$$E(Y_t|Y_{t-1} = y_{t-1}, \ldots, Y_0 = y_0; X_{t-1} = x_{t-1}, \ldots, X_0 = x_0),$$

where the small letters refer to realized values of the stochastic variables. As time passes, more and more realizations of the exogenous and endogenous variables become known. The information thus *expands* with rising t.

In mathematical statistics a precise definition of the conditioning information set can be given such that I_{t-1} appears as a clearly demarcated *set* (in the formal sense) with the *inclusion property* $\ldots I_{t-1} \subseteq I_t \subseteq I_{t+1} \ldots$. This inclusion property reflects that "more and more is known to have happened". Expounding the precise definition of information sets requires a formal conceptual apparatus beyond the scope of this text. For our purposes an intuitive notion of information will suffice. The key feature is that an *expanding* information set means that more and more ex ante possible "states of the world" can be ruled out. The other side of the coin is that the set of possible states of the world *shrinks* over time. Indeed, as more information becomes available, more and more "uncertainty" is resolved. An increasing amount of information and reduced uncertainty are thus two sides of the same thing.[2]

For a simple example of a model-consistent expectation, consider a model of type A combined with myopic expectations and $X_t = \bar{x} + \varepsilon_t$. On the basis of (25.9), we find the model-consistent expectation to be

$$E(Y_t|I_{t-1}) = aY_{t-1} + c\bar{x}.$$

As another example, consider a model of type B, again with myopic expectations and $X_t = \bar{x} + \varepsilon_t$. From (25.12), we get

$$E(Y_{t+1}|I_t) = c\frac{\bar{x}}{1-a}.$$

Now to *rational* expectations.

25.2 The rational expectations hypothesis

Unsatisfied with mechanistic formulas for agents' subjective expectations like those in Section 25.1.1, the American economist John F. Muth (1961) introduced a radically different approach, the hypothesis of *rational expectations.* Muth stated the hypothesis the following way:

> I should like to suggest that expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant economic theory. At the risk of confusing this purely descriptive hypothesis with a pronouncement as to what firms ought to do, we call such expectations 'rational' (Muth 1961).

[2] Appendix A is a refresher on conditional expectations and a warning against a potential confusion regarding the inclusion relationship between conditioning sets appearing to the right of the separator "|". The inclusion relationship between information sets differs from that between the corresponding subsets of the outcome space for the conditioning variables.

Muth applied this hypothesis to simple microeconomic problems with stochastic elements. The hypothesis was subsequently extended and applied to general equilibrium theory and macroeconomics. Nobel laureate Robert E. Lucas from the University of Chicago lead the way by a series of papers starting with Lucas (1972), Lucas (1973), and Lucas (1975). To assume agents hold rational expectations instead of, for instance, adaptive expectations may radically alter the results, including the impact of economic policy, both quantitatively and qualitatively. This lead to a profound change in macroeconomists' thinking, the "rational expectations revolution" of the 1970s.

25.2.1 The concept of a rational expectation

Assuming the economic agents hold *rational expectations* is to assume that their *subjective expectation equals the model-consistent expectation.* As we have detailed in the previous section, the latter is the mathematical conditional expectation that can be calculated on the basis of the model and available relevant information about the exogenous stochastic variables. So, the hypothesis of rational expectations is the hypothesis that

$$Y_{t-1,t}^{e} = E(Y_t|I_{t-1}), \tag{25.15}$$

saying that agents' subjective conditional expectation coincides with the "true" conditional expectation, given the model in question, that is, the mathematical *model-consistent conditional expectation.* The agents are thus assumed not to make systematic forecast errors.

If our point of departure is a model of type A, combining it with the rational expectations hypothesis implies that we can write the model in compact form as

$$Y_t = aE(Y_t|I_{t-1}) + c\, X_t, \qquad t = 1, 2, ... \tag{25.16}$$

This equation makes up a simple *rational expectations model* (henceforth an RE model), where past expectations of current endogenous variables affect these. As a simple example, consider:

EXAMPLE 2 In the model (25.7), let $a = 0$ so that $Y_t = cX_t$, and assume that the process $\{X_t\}$ is a random walk, $X_t = X_{t-1} + \varepsilon_t$. Then the myopic expectation of Y_t as seen from period $t-1$ is $Y_{t-1,t}^{e} = Y_{t-1}$, cf. (25.2). The rational expectation of Y_t as seen from period $t-1$ is $E(Y_t|I_{t-1}) = cE(X_t|I_{t-1}) = cX_{t-1} = Y_{t-1}$. In this example, the myopic expectation is thus also the rational expectation. □

If our point of departure is a model of type B, combining it with rational expectations gives, in compact form,

$$Y_t = aE(Y_{t+1}|I_t) + c\, X_t, \qquad t = 0, 1, 2, ... \tag{25.17}$$

This equation makes up an RE model, where current expectations of the future value of the endogenous variables affect the current values of these – in brief, an RE model with forward-looking rational expectations. This model framework will be our main focus in the next chapter.

Returning to model type A, but in contrast to Example 2, we shall now open up for $a \neq 0$.

Solving an RE model with past expectations of current endogenous variables

To solve a stochastic model means to find the stochastic process followed by the endogenous variable(s), Y_t, given the stochastic process followed by the exogenous variable(s), X_t. For a linear RE model with past expectations of current endogenous variables, the solution procedure is the following.

1. By substitution, reduce the RE model (or the relevant part of the model) into a form like (25.16) expressing the endogenous variable in period t in terms of its past expectation and the exogenous variable(s). (The case with multiple endogenous variables is treated similarly.)

2. Take the conditional expectation on both sides of the equation and solve for the conditional expectation of the endogenous variable.[3]

3. Insert into the "reduced form" attained at 1.

In practice there is often a fourth step, namely to express *other* endogenous variables in the model in terms of those found in step 3. Let us see how the procedure works by way of the following example.

EXAMPLE 3 *(imported raw materials and domestic price level under rational expectations)* We modify Example 1 by replacing myopic expectations by rational expectations, i.e., $P^e_{t-1,t} = E(P_t|I_{t-1})$. We still assume $X_t = \bar{x} + \varepsilon_t$. *Step 1:*

$$P_t = aE(P_t|I_{t-1}) + c\ X_t, \qquad 0 < \alpha < 1, c > 0, \quad t = 1, 2, ... \tag{25.18}$$

Step 2: $E(P_t|I_{t-1}) = aE(P_t|I_{t-1}) + c\bar{x}$, implying

$$E(P_t\,|I_{t-1}) = \frac{c\bar{x}}{1-a}.$$

[3] It is here assumed that the model is not degenerate. In the model (25.16), this requires $a \neq 1$. If $a = 1$, the model is inconsistent unless $E(X_t\,|I_{t-1})) = 0$ in which case there are infinitely many solutions. Indeed, for any number $k \in (-\infty, +\infty)$, the process $Y_t = k + cX_t$ solves the model when $E(X_t\,|I_{t-1}) = 0$.

Step 3: Insert into (25.18) to get

$$P_t = a\frac{c\bar{x}}{1-a} + c(\bar{x} + \varepsilon_t) = P^* + c\varepsilon_t, \qquad t = 1, 2, ...$$

where $P^* \equiv c\bar{x}/(1-a)$. So $E(P_t \,|I_{t-1}) = P^*$ for $t = 1, 2, ...$ The structure and the parameters may have been different before period 0, and so we take the expectation of P_0 as seen for period -1 as given. Thus P_0 is fixed, given the disturbance ε_0. We see that under rational expectations, the economy functions such that a deviation of this P_0 from P^* has no impact on the price from period 1 and onward. For $t = 1, 2, ...,$ the price equals its constant expected value plus the noise term. The forecast error, $P_t - E(P_t \,|I_{t-1})$, has zero mean. These are important differences compared with the myopic expectations in Example 1. □

We return to the general form (25.16). Before specifying the process $\{X_t\}$, the second step gives

$$E(Y_t \,|I_{t-1}) = c\frac{E(X_t \,|I_{t-1})}{1-a}, \tag{25.19}$$

presupposing $a \neq 1$. Then, in the third step we get

$$Y_t = c\frac{aE(X_t \,|I_{t-1}) + (1-a)X_t}{1-a} = c\frac{X_t - a(X_t - E(X_t \,|I_{t-1}))}{1-a}. \tag{25.20}$$

Let X_t follow the process $X_t = \bar{x} + \rho X_{t-1} + \varepsilon_t$, where $0 < \rho < 1$ and ε_t has zero expected value, given all observed past values of X and Y. Then (25.20) yields the solution

$$Y_t = c\frac{X_t - a\varepsilon_t}{1-a} = c\frac{\bar{x} + \rho X_{t-1} + (1-a)\varepsilon_t}{1-a}, \qquad t = 0, 1, 2, ...$$

In Exercise 2 you are asked to solve a simple Keynesian model of this form and compare the solution under rational expectations with the solution under myopic expectations.

Expounding the concept of rational expectations

Assuming rational expectations means assuming that the economic actors do not make systematic expectation errors. This assumption is often convenient, but a drastic simplification that at best offers an approximation. *First*, the assumption entails that the economic actors[4] share one and the same understanding about how the economic system functions (and in this chapter they also share one and

[4] Or, to be more precise, the economic actors whose expectations matter for the aggregate utcome Y_t.

the same information, I_{t-1}). This is already a big mouthful. *Second*, this understanding is assumed to *comply with the model* put forward by the informed economic specialist. *Third*, that model is supposed to be an accurate ("true") model of the economic process (otherwise the actors *would* make systematic expectation errors and gradually experience this). The actors' supposed knowledge not only embraces the accurate model structure, but also its "true" parameter values as well as the parameter values of the stochastic process which X_t follows. Indeed, by equalizing $Y^e_{t-1,t}$ with the true conditional expectation, $E(Y_t|I_{t-1})$ in (25.16), rather than with some econometric estimate of this, it is presumed that the actors know the exact values of the parameters a and c in the data-generating process which the model is supposed to mimic. In practice it is not possible to attain such precise knowledge, at least not unless the considered economic system has reached a steady state and no structural changes occur. This condition is hardly ever satisfied in macroeconomics.

Nevertheless, a model based on the rational expectations hypothesis can in many contexts be seen as a useful cultivation of a theoretical research question. The results that emerge cannot be due to *systematic* expectation errors from the economic agents' side. In this sense the assumption of rational expectations makes up a theoretically interesting *benchmark case*. On the other hand, there are issues, in particular related to business cycles, where systematic expectation errors – say excess optimism or pessimism – are a key ingredient of the phenomenon to be studied. Then the assumption of rational expectations would of course be a bad point of departure.

Finally, a terminological remark. As witnessed by the reservation made by Muth (2001) in the quotation above, the term "rational expectations" itself is not unproblematic. Usually, in economists' terminology, "rational" refers to behavior based on optimization subject to the constraints faced by the agent. So one might think that the RE hypothesis stipulates that economic agents try to get the most out of a situation with limited information, contemplating the benefits and costs of gathering more information and using more elaborate statistical estimation methods. But this is a misunderstanding. The RE hypothesis presumes that an essentially "correct" model of the system is already known to the agents. The "rationality" just refers to taking this assumed knowledge fully into account in the chosen actions.

Anyway, the term "rational expectations" has become standard and we shall stick to it.

25.2.2 The forecast error*

Let the forecast of some variable Y one period ahead be denoted $Y^e_{t-1,t}$. Suppose the forecast is determined by a given function, f , of realizations of Y and X up to and including period $t-1$, that is, $Y^e_{t-1,t} = f(y_{t-1}, y_{t-2}, ..., x_{t-1}, x_{t-2}, ...)$. Such a function is known as a *forecast function.* It might for instance be one of the mechanistic forecasting principles in Section 25.1.1. At the other extreme the forecast function might coincide, at least theoretically, with the model-consistent conditional expectation. In the latter case it is a *model-consistent forecast function* and we can write

$$\begin{aligned} Y^e_{t-1,t} &= f(y_{t-1}, y_{t-2}, ..., x_{t-1}, x_{t-2}, ...) = E(Y_t \,| I_{t-1}) \qquad (25.21) \\ &= E(Y_t \,| Y_{t-1} = y_{t-1}, Y_{t-2} = y_{t-2}, ..., x_{t-1} = x_{t-1}, x_{t-2} = x_{t-2}, ...) \,. \end{aligned}$$

The *forecast error* is the difference between the actual value, Y_t, of a variable and the in advance forecasted value. So, for a given forecast, $Y^e_{t-1,t}$, the *forecast error* is $e_t \equiv Y_t - Y^e_{t-1,t}$ and is itself a stochastic variable.

If the forecast function in (25.21) complies with the true data-generating process (a big "if"), then the implied forecasts would have several ideal properties:

(a) the forecast error has zero mean;

(b) the forecast error is uncorrelated with any of the variables in the information set I_{t-1} and therefore also with its own past values; and

(c) the expected squared forecast error is minimized.

To see this, note that the model-consistent forecast error is $e_t = Y_t - E(Y_t \,| I_{t-1})$. From this follows that $E(e_t \,| I_{t-1}) = 0$, cf. (a). Also the unconditional expectation is nil, i.e., $E(e_t) = 0$. This is because $E(E(e_t \,| I_{t-1})) = E(0) = 0$ at the same time as $E(E(e_t \,| I_{t-1})) = E(e_t)$. The latter equality follows by the *law of iterated expectations* from statistics which says that the unconditional expectation of the conditional expectation of a stochastic variable Z is given by the unconditional expectation of Z, cf. Appendix A. Indeed, if beforehand we have good reasons to expect that we will revise our expectations upward, say, if we receive additional information, then the original expectation must be biased, hence not rational.[5]. Considering the specific model (25.7), the model-consistent-forecast error is $e_t = Y_t - E(Y_t \,| I_{t-1}) = c(X_t - E(X_t \,| I_{t-1}))$, by (25.19) and (25.20). An ex post

[5] Imagine you ask a stockbroker in which direction she expects to revise her expectations upon the arrival of more information. Suppose the broker answers "upward", say. Well, another broker would then be recommendable.

error ($e_t \neq 0$) thus emerges if and only if the realization of the exogenous variable deviates from its conditional expectation as seen from the previous period.

As to property (b), for $i = 1, 2, ...$, let s_{t-i} be some variable value belonging to the information I_{t-i}. Then, property (b) is the claim that the (unconditional) covariance between e_t and s_{t-i} is zero, i.e., $\text{Cov}(e_t s_{t-i}) = 0$, for $i = 1, 2, ...$. This follows from the *orthogonality property* of model-consistent expectations (see Appendix B). In particular, with $s_{t-i} = e_{t-i}$, we get $\text{Cov}(e_t e_{t-i}) = 0$, i.e., the forecast errors exhibit *lack of serial correlation.* If the covariance were not zero, it would be possible to improve the forecast by incorporating the correlation into the forecast. In other words, under the assumption of rational expectations economic agents have no more to learn from past forecast errors. As remarked above, the RE hypothesis precisely refers to the fictional situation where learning has been completed and underlying mechanisms do not change.

Finally, a desirable property of a forecast function $f(\cdot)$ is that it maximizes "accuracy", i.e., minimizes an appropriate loss function. A popular loss function, L, in this context is the expected squared forecast error conditional on the information I_{t-1},

$$L = E((Y_t - f(y_{t-1}, y_{t-2}, ..., x_{t-1}, x_{t-2}, ...))^2 \,| I_{t-1}) .$$

Assuming $Y_t, Y_{t-1}, ..., X_{t-1}, X_{t-2}, ...$ are jointly normally distributed, then the solution to the problem of minimizing L is to set $f(\cdot)$ equal to the conditional expectation $E(Y_t \,| I_{t-1})$ based on the data-generating model as in (25.21).[6] This is what property (c) refers to.

EXAMPLE 4 Let $Y_t = aE(Y_t \,| I_{t-1}) + cX_t$, with $X_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and ε_t is white noise with variance σ^2. Then (25.20) applies, so that

$$Y_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t, \qquad t = 0, 1, ...,$$

with variance $c^2\sigma^2$. The model-consistent forecast error is $e_t = Y_t - E(Y_t \,| I_{t-1}) = c\varepsilon_t$ with conditional expectation equal to $E(c\varepsilon_t \,| I_{t-1}) = 0$. This forecast error itself is white noise and is therefore uncorrelated with the information on which the forecast is based. □

It is worth emphasizing that in practice the "true" conditional expectation usually can not be known – neither to the economic agents nor to the investigator; sometimes it does not even exist due to the presence of *fundamental*

[6]For proof, see Pesaran (1987). Under the restriction of only *linear* forecast functions, property (c) holds even without the joint normality assumption, see Sargent (1979).

uncertainty.[7] At best there can be a reasonable estimate, probably somewhat different across the agents because of differences in information and conceptions of how the economic system functions. Deeper models of expectations attempt to give an account of the way agents *learn* about the economic environment. An important ingredient here is how agents contemplate the costs and potential gains associated with further information search needed to reduce systematic expectation errors where possible. This contemplation is intricate because information search often means entering unknown territory. Moreover, for a large subset of the agents the costs may be prohibitive. A further complicating factor involved in learning is that when the agents have obtained *some* knowledge about the statistical properties of the economic variables, the resulting behavior of the agents tends to *modify* these statistical properties. This kind of modeling is complicated and belongs to the current research frontier.

The rational expectations hypothesis sets these problems aside. It is simply assumed that the learning process has been completed and the structure of the economy remains unchanged.

25.2.3 Perfect foresight as a special case

The notion of *perfect foresight* corresponds to the limiting case where the variance of the exogenous variable(s) is zero so that with probability one, $X_t = E(X_t | I_{t-1})$ for all t. Then we have a non-stochastic model where rational expectations imply that agents' ex post forecast error with respect to Y_t is zero.[8] To put it differently: rational expectations in a non-stochastic model is equivalent to perfect foresight. Note, however, that perfect foresight necessitates the exogenous variable X_t to be known in advance. Many real-world situations are not like that. If we want our model to take this into account, the model ought to be formulated in an explicit stochastic framework. And assumptions should be stated about how the economic agents respond to the uncertainty in their behavior and with respect to expectations formation. The rational expectations assumption is an approach to the latter problem. It has been much applied in macroeconomics since the early 1980s, perhaps because it at least helps providing insight in an important benchmark case.

The remainder of this chapter addresses the issue of money neutrality or non-neutrality in a framework where agents have rational expectations.

[7] *Fundamental uncertainty* is present in situations where the full "range" of possible outcomes is not even known, hence cannot be endowed with a probability ("it is not known what is unknown"). In Section 26.2.4 of the next chapter we shall have a little more to say about this situation.

[8] Here we disregard zero probability events.

25.3 Wage setting in advance

The specific economic question to be studied in the subsequent sections is:

> Do rational expectations rule out persistent effects on output of changes in money supply?

Effects of money supply shocks are said to be *persistent* to the extent that the real effects go beyond the period in which the shock occurs. As documented in surveys by Blanchard (1990) and Stock and Watson (1999), the empirics indicate that there is definitely a tendency for monetary shocks to have quite persistent effects on output. On the other hand, the effects on output of even permanent shifts in the money supply tend to gradually die out.

We shall address some suggested explanations of these phenomena, based on a series of simple *stochastic aggregate demand-aggregate supply (AD-AS)* models for a closed economy where agents have rational expectations. The models are stated in log-linear form. Then one can apply the simple rule that the expected value of a sum is the sum of the expected values. There are many cases where linearity in logs seems an acceptable first approximation (cf. Appendix C).

The AD-AS models to be considered are dynamic short-run models with both a Keynesian and a monetarist flavour. The "Keynesian" flavor comes from the presence of nominal rigidities, the "monetarist" flavor from the assumption of an autonomous money stock.

Some (unspecified) kind of imperfect competition in both the labor market and the output market is assumed. The nominal wage is in every period predetermined, by monopolist trade unions or through bargaining in advance. Behind the scene there are various kinds of contracting costs making it advantageous to contract for fixed periods of time and renegotiate when the contract is about to expire.

It turns out that the degree of persistence of the effects of monetary shocks depends very much on the *timing* of the wage contracts. First, it matters whether wages are negotiated at the different local levels (where "local" may refer to a particular firm or a particular craft union) at the *same* time (synchronous wage setting) or in some kind of *staggered* pattern (asynchronous wage setting). As we shall see, in a RE model only in the last case will there be persistent effects of changes in changes in the money supply. Second, in the case of staggered wage setting, it matters whether wages, negotiated for two periods, say, can be set at *different levels* over the two periods or not. For example, wage contracts may specify that the wage in the second period should be α per cent above the level in the first period. Alternatively, wages might have to be set at the *same level* over the two periods. In that case, which is natural when the period length is relatively short, pronounced persistence of real effects of changes in money supply arises under rational expectations.

The two models to be applied for throwing light on these themes were formulated by the American economists Stanley Fischer (1977) and John Taylor (1979, 1980), respectively. In the stylized versions discussed here, both models consider wages to be set in advance for *two* periods. In Fischer's model there is staggered wage setting with possibly different wage levels in the two periods. In Taylor's model there is staggered wage setting with the same wage level for the two periods. This difference is the key to understanding why Fischer's model does not predict persistent real effects of money while Taylor's does.

To put the notion of staggering clearly into relief we start out from a *benchmark model* which has *synchronous* wage setting. Throughout the focus is on how (if at all) anticipated and unanticipated money supply changes affect output under the conditions of *wage sluggishness* in the sense of pre-determined wages.

25.4 A benchmark model with synchronous wage setting

We follow the general convention in macroeconomic log-linear analysis that variables measured in natural units are denoted by capital letters and the logarithm of such a variable is denoted by the corresponding lower case letter.[9] These lower case letters are then the stochastic variables that are assumed related linearly. So from now on we depart from the convention in mathematical statistics using capital letters for stochastic variables and lower case letters for the specific values (real numbers) these can take (as in the exposition above). It should be clear from the context whether a given letter should be interpreted as a stochastic variable or as a specific value of this variable.

A further notational simplification is that from now on the expected value of any stochastic variable, z_t, conditional on information I_{t-1}, will be denoted $E_{t-1}z_t$. So the RE hypothesis is from now written

$$z^e_{t-1,t} = E_{t-1}z_t \equiv E(z_t|I_{t-1}). \tag{25.22}$$

[9] Usually the interest rate is an exception in that i usually means the nominal interest rate, not its logarithm. But in this chapter, the interest rate is not a key variable anyway.

25.4.1 Wage setting one period in advance

We consider a dynamic AD-AS model with synchronous wage setting one period in advance. The model consists of these three equations:

$$
\begin{aligned}
y_t &= m_t - p_t, && (25.23)\\
y_t &= -(w_t - p_t), && (25.24)\\
w_t &= w_{t-1,t} = E_{t-1}p_t, && (25.25)
\end{aligned}
$$

where y_t is output, m_t the nominal money supply, p_t is the price level, and w_t is the nominal wage, all in logs and referring to period t. Further, $w_{t-1,t}$ is the wage set in period $t-1$ for period t, and $E_{t-1}p_t$ is the expected price level in period t, conditional on information available in period $t-1$. We will often speak of y, p, and w as just "output", "price level" and "wage level" (instead of "log of output" etc.). The money supply, m_t, is treated as an exogenous stochastic variable.

Equation (25.23) is the *aggregate demand* equation (AD equation), equation (25.24) is the *aggregate supply* equation (AS equation), and (25.25) is the *wage setting* equation. We now provide detailed interpretations of these three equations.

The AD equation

The reduced form of a simple IS-LM framework with exogenous money supply says that output, y, equals output demand, the latter depending on money supply, the current price level, *and* the expected price level next period. In (25.23), however, the expected price level next period does not enter. This reflects a simplifying assumption that the quantity theory of money holds approximately (velocity of money being independent of the nominal interest rate).

Let us spell this out in more detail. Let i be the nominal interest rate and π^e the expected forward-looking inflation rate, i.e., $\pi^e \equiv (P^e_{+1} - P)/P \approx p^e_{+1} - p$. In a customary IS-LM model one considers an IS equation like

$$
Y = D(Y, r) \cdot U^{IS}, \qquad 0 < D_Y < 1, D_r < 0, \tag{25.26}
$$

where U^{IS} is a multiplicative demand disturbance and $r = (1 + i)/(1 + \pi^e) - 1$ $\approx i - (p^e_{+1} - p)$ is the (expected) real interest rate. We suppress the dating of the variables where not needed. The equation (25.26) defines Y as an implicit function, written $Y = \tilde{Y}(r, U^{IS})$, of r and U^{IS}. A log-linear approximation gives $y = c_0 - c_1(i - (p^e_{+1} - p)) + c_2u^{IS}$. For simplicity we set $c_0 = 0$ and $c_2 = 1$, so that

$$
y = -c_1(i - (p^e_{+1} - p)) + u^{IS}, \qquad c_1 > 0, \tag{IS}
$$

Similarly, the customary LM equation is

$$\frac{M}{P} = L(Y,i) \cdot U^{LM}, \qquad L_Y > 0, L_i < 0,$$

where U^{LM} is a multiplicative money demand disturbance, by Keynesians commonly called a liquidity preference shock, by monetarists a velocity shock. A log-linear approximation is $m - p = b_0 + b_1 y - b_2 i + u^{LM}$, where we simplify by setting $b_0 = 0$. Thus,

$$m - p = b_1 y - b_2 i + u^{LM}, \qquad b_1 > 0, b_2 > 0. \tag{LM}$$

Isolating i and substituting into (IS), we end up with

$$y = \frac{c_1}{b_2 + b_1 c_1}(m - u^{LM} - p) + \frac{b_2 c_1}{b_2 + b_1 c_1}(p^e_{+1} - p) + \frac{b_2}{b_2 + b_1 c_1} u^{IS}.$$

In the simple case of the quantity theory of money, we have $b_2 = 0$. Then the output demand disturbance u^{IS} as well as expected inflation, $p^e_{+1} - p$, is uncoupled from the system. Being interested only in qualitative properties, we simplify by setting $b_1 = 1$ so as to get $y = m - u^{LM} - p$. Since both m and $-u^{LM}$ are exogenous and only occur together and additively, we can shorten notation by considering only one exogenous variable, $\tilde{m} \equiv m - u^{LM}$. We rename this variable as m and the reader is free to interpret this as if $u^{LM} \equiv 0$. Thus we finally arrive at (25.23). When needed, we shall re-introduce u^{LM} explicitly as a separate component.

Although the quantity theory is unrealistic, it helps us sidestep the technical issues of *forward-looking* expectations associated with terms like $p^e_{+1} - p$.[10] These are not of key importance in a first approach to the question at hand. All in all, (25.23) is merely a reordering of a money market clearing condition saying that real money supply equals real money demand, which is simplifying assumed interest inelastic.

The AS equation

The aggregate supply equation, (25.24), reflects firms' profit maximization. If $Y = L^a K^{1-a}$, $0 < a < 1$, where K denotes capital input, then the marginal productivity of labor is $\partial Y/\partial L = aL^{a-1}K^{1-a}$. Since K moves little in the short run, we fix it at the value 1. Then marginal cost is $MC = W/(aL^{a-1})$. Suppose there is monopolistic competition and that the price elasticity of demand is a constant, $-\eta < 0$. Then profit maximizing firms choose a constant mark-up μ

[10] This modeling trick became popular after Lucas (1973). It considerably increases the tractability of the stocastic AD-AS model which otherwise, as witnessed by Chapter 27, is analytically complex.

$= \eta/(\eta-1) > 1$ so that $P = \mu MC = \mu W/(aL^{a-1})$; perfect competition can be interpreted as the limiting case $\eta = \infty$ whereby $\mu = 1$.

We now have $L = [(\mu/a)W/P]^{1/(a-1)}$ so that

$$Y = L^a = [(\mu/a)W/P]^{a/(a-1)}. \tag{25.27}$$

By taking logs on both sides and ignoring unimportant constants, we get (25.24).[11] Note that, unless $\mu = 1$ (the case of perfect competition), (25.27) is not an aggregate supply function in the classical sense of a relation determining the amount firms will supply when facing a given price, P, and given input prices. Beyond perfect competition, firms are price makers and not price takers. Then (25.27) should be seen as just stating what combinations of P and Y are consistent with firms' profit maximization, given the wage level W. This broad meaning is understood when we use the convenient term "AS equation".

The wage setting equation

Finally, the interpretation of the wage setting equation, (25.25), is that workers set or negotiate wages so as to achieve, in expected value, a target real wage approximately equal to 1. If the price level turns out as expected, $p_t = E_{t-1}p_t$, then, according to (25.25), $w_t - p_t = 0$, i.e., $\ln(W_t/P_t) = 0$ or $W_t/P_t = 1$. Nonetheless, strictly speaking, (25.25) implies a target real wage, ω, slightly above 1. Indeed, letting ω denote the target real wage (thus also the expected real wage), we have $\omega = E_{t-1}(W_t/P_t) = W_t E_{t-1}(P_t^{-1})$, so that

$$\ln \omega = \ln W_t + \ln E_{t-1}(P_t^{-1}) > \ln W_t + E_{t-1}\ln(P_t^{-1}) = w_t - E_{t-1}p_t = 0,$$

where the inequality follows from Jensen's inequality for a strictly concave function (the ln function) and where the last equality follows from the wage setting behavior (25.25). This implies $\omega > 1$, but the difference is small, if the variance of $1/P_t$ is small so that $E_{t-1}\ln(1/P_t)$ approximates $\ln E_{t-1}(1/P_t)$ reasonably well.

25.4.2 Solving the benchmark model

The only exogenous variable is m_t which follows a stochastic process known to the agents. The agents are assumed to have the information needed to calculate the true conditional expectation of m_t as seen from the end of period $t-1$, that is, $E_{t-1}m_t$. The remaining four variables, y_t, p_t, w_t, and $w_{t-1,t}$, are endogenous. And we have four equations, since (25.25) contain two equations. This seems promising.

We follow the solution procedure described in Section **??**.

[11] Since we are interested only in qualitative properties, additive constants are ignored and unimportant positive coefficients put equal to 1, which here corresponds to setting $a = \frac{1}{2}$.

1. The endogenous variable the expectation of which appears in the model is p_t. Inserting (25.25) into (25.24) gives $y_t = p_t - E_{t-1}p_t$. Substituting this into (25.23) gives $p_t - E_{t-1}p_t = m_t - p_t$ or
$$2p_t = E_{t-1}p_t + m_t. \tag{25.28}$$

2. Taking the conditional expectation on both sides gives $2E_{t-1}p_t = E_{t-1}p_t + E_{t-1}m_t$. Thus the conditional expectation of p is found to be
$$E_{t-1}p_t = E_{t-1}m_t. \tag{25.29}$$

3. Inserting into the reduced form (25.28) yields $p_t = \frac{1}{2}(E_{t-1}m_t + m_t)$, i.e.,
$$p_t = \frac{1}{2}m_t - E_{t-1}m_t) + E_{t-1}m_t. \tag{25.30}$$

4. Given this solution for the realized value of the price level in period t, the realized value of output is found by substituting (25.30) into (25.23), which gives
$$y_t = \frac{1}{2}(m_t - E_{t-1}m_t). \tag{25.31}$$
Finally, substituting (25.29) into (25.25) gives the solution for the nominal wage,
$$w_t = w_{t-1,t} = E_{t-1}m_t. \tag{25.32}$$

We see from (25.31) that with *synchronous wage setting* only the *unanticipated* part of money supply affects output. This is similar to the *policy ineffectiveness* conclusion from Lucas (1972, 1973).[12] The mechanism is slightly different, however, because we consider an AD-AS model where the nominal wage is set in advance. When positive "surprises" in money supply occur, wages are already set, but prices can adjust, cf. (25.30), thus allowing an equilibrium with lower *ex post* real wage and higher labor demand and employment, hence also higher output.

We have ignored unimportant constants in such a way that the obvious interpretation of the model's y_t is as an *output gap.* That is, we may take y_t to represent $\ln Y_t = \ln Q_t - \ln \bar{Q}_t$, where Q_t is actual aggregate output (GDP) and $\bar{Q}_t$ is the "natural" level of output (that level which obtains when expectations are realized).

The conclusion is that with synchronous wage setting, money supply only affect output within the time interval for which wages are preset. In the present

[12] See Chapter 27.

case wages were reset every period in advance, and then the effect of a money supply shock lasts only one period. Assuming policy makers can not act more often − and do not have more information − than wage-setters, policy makers can not stabilize output. If wages were preset synchronously for *two* periods and policy makers could act every period, there would be *some* scope for stabilization policy. The important thing to notice is that with synchronous wage setting there are, under rational expectations, no lasting effects of shocks.

25.5 Asynchronous wage setting for several periods: Fischer's approach

The fundamental characteristic of Fischer's approach is that wages are set in an asynchronous way for two periods with possibly different levels in the two periods. One half of the labor force, group A, presets at the end of period $t-1$ its nominal wages for period t and period $t+1$, respectively. Thereby $w_{t-1,t}$ and $w_{t-1,t+1}$ are fixed. At the end of period $t+1$, group A resets wages for the next two periods and so on. The other half, group B, has at the end of period $t-2$ preset its nominal wages for period $t-1$ and period t, respectively, i.e., $w_{t-2,t-1}$ and $w_{t-2,t}$. At the end of period t, group B resets wages for the next two periods and so on. Fig. 25.1 illustrates.[13] Each group sets wages for each period with the aim of achieving, in expected value, the target real wage, which we assume to be constant and approximately equal to one. A possible explanation of the existence of staggering is that the economic agents, for example trade unions, want to obtain information about what is going on in other markets before decisions are made.

As mentioned above, Fischer's model does not predict persistent real effects of money while Taylor's does. The two models also differ in other respects that may be of interest. We will therefore present the Fischer model in two versions. The first version is essentially the same as his original model, whereas the second is close to Taylor's setup except in one key respect. This will make it easier to identify the fundamental source of persistence.

As in the benchmark model above, in the first version of Fischer's model only the wage level is pre-determined. The price level is assumed flexible and formed through a constant mark-up on rising marginal production costs *in* the period. In Taylor's setup, however, marginal production costs are scale independent and then the constant mark-up implies that also the price is effectively pre-determined when the wage is. But now to Fischer's original model.

[13] Following our usual timing convention, period t is the time interval $[t, t+1)$.

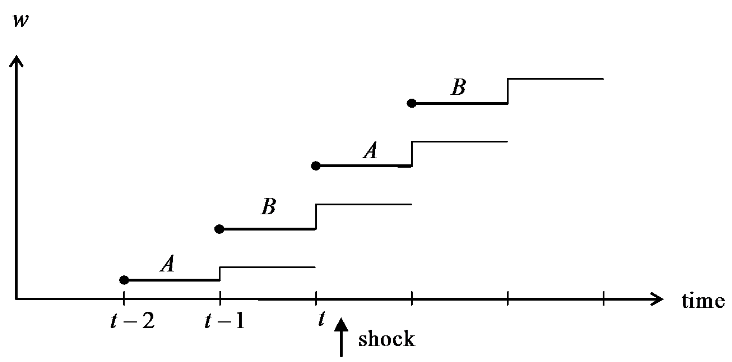

Figure 25.1: Configuration of wage setting in the Fischer models.

25.5.1 The original Fischer model

The model is:

$$\begin{aligned} y_t &= m_t - p_t, && (25.33) \\ y_t &= -(w_t - p_t) = -\left[\frac{1}{2}(w_{t-1,t} + w_{t-2,t}) - p_t\right], && (25.34) \\ w_{t-1,t} &= E_{t-1}p_t, \qquad w_{t-2,t} = E_{t-2}p_t. && (25.35) \end{aligned}$$

The "general" wage level w_t in period t is now an arithmetic average of (the log of) wages set by different groups in period $t-1$ and period $t-2$, respectively. The general wage level may also be written $w_t \equiv \ln W_t \equiv \ln\left[(W_{t-1,t} \cdot W_{t-2,t})^{1/2}\right]$, so that w_t is the log of the geometric average of the actual wage levels, $W_{t-1,t}$ and $W_{t-2,t}$, set in advance by the two groups for period t. Similarly, p should be interpreted as an average price level. We will call this version of Fischer's contribution *version I.*

We follow our standard solution procedure:

1. Inserting (25.35) into (25.34) gives $y_t = -(\frac{1}{2})(E_{t-1}p_t + E_{t-2}p_t) + p_t$. Substituting this into (25.33) yields

$$2p_t = \frac{1}{2}(E_{t-1}p_t + E_{t-2}p_t) + m_t. \qquad (25.36)$$

2. We then take the conditional expectation as seen from $t-2$ on both sides to get

$$\begin{aligned} 2E_{t-2}p_t &= \frac{1}{2}\left[E_{t-2}(E_{t-1}p_t) + E_{t-2}p_t\right] + E_{t-2}m_t \\ &= E_{t-2}p_t + E_{t-2}m_t, \quad \text{so that} \\ E_{t-2}p_t &= E_{t-2}m_t, && (25.37) \end{aligned}$$

where we have used that $E_{t-2}(E_{t-1}p_t) = E_{t-2}p_t$, according to the *law of iterated expectations* (cf. Appendix A). In words: your best forecast today of how you are going to forecast tomorrow the price the day after tomorrow, will be the same as your best forecast today of the price the day after tomorrow; *it takes new information to change a rational expectation.* Second, we take the conditional expectation as seen from $t-1$ on both sides of (25.36) to get $2E_{t-1}p_t = (\frac{1}{2})(E_{t-1}p_t + E_{t-2}p_t) + E_{t-1}m_t$. Substituting (25.37) into this and ordering yields

$$E_{t-1}p_t = \frac{1}{3}E_{t-2}m_t + \frac{2}{3}E_{t-1}m_t. \tag{25.38}$$

3. To find the solution for p_t, substitute (25.37) and (25.38) into (25.36) and reorder:

$$\begin{aligned} p_t &= \frac{1}{2}(m_t + \frac{1}{3}E_{t-1}m_t + \frac{2}{3}E_{t-2}m_t) \qquad (25.39)\\ &= \frac{1}{2}(m_t - E_{t-1}m_t) + \frac{2}{3}(E_{t-1}m_t - E_{t-2}m_t) + E_{t-2}m_t. \end{aligned}$$

4. The solution for y_t is found by inserting this into (25.33) to get

$$\begin{aligned} y_t &= \frac{1}{2}\left(m_t - \frac{1}{3}E_{t-1}m_t - \frac{2}{3}E_{t-2}m_t\right) \\ &= \frac{1}{2}(m_t - E_{t-1}m_t) + \frac{1}{3}(E_{t-1}m_t - E_{t-2}m_t) \\ &= \frac{1}{3}\left[\frac{1}{2}(m_t - E_{t-1}m_t) + m_t - E_{t-2}m_t\right]. \qquad (25.40) \end{aligned}$$

We have expressed the solutions for p_t and y_t, respectively, in several distinct ways in order to ease interpretation. The first equalities in (25.39) and (25.40) reveal that the effects on p_t and y_t, respectively, of previously formed expectations have opposite sign and exhibit symmetry in the sense that there is no effect on $p_t + y_t$. This is as expected in view of (25.33), which implies $p_t + y_t = m_t$.[14] The second equalities in (25.39) and (25.40) reveal that the effects on p_t and y_t of shocks in period t and period $t-1$, respectively, have the same sign and sum to $1\times$ (size of shock). Again no surprise, in view of $p_t + y_t = m_t$. Fig. 25.2 illustrates the effect of a positive monetary shock in period t. For given expectations, w_t is fixed and this defines the position of the upward-sloping AS curve. The position

[14] The additional symmetry displayed by the coefficients 1/2, 1/3, etc., is an artifact due to the simplifying choice of parameter values such that the coefficients in the AD and AS relations become unity.

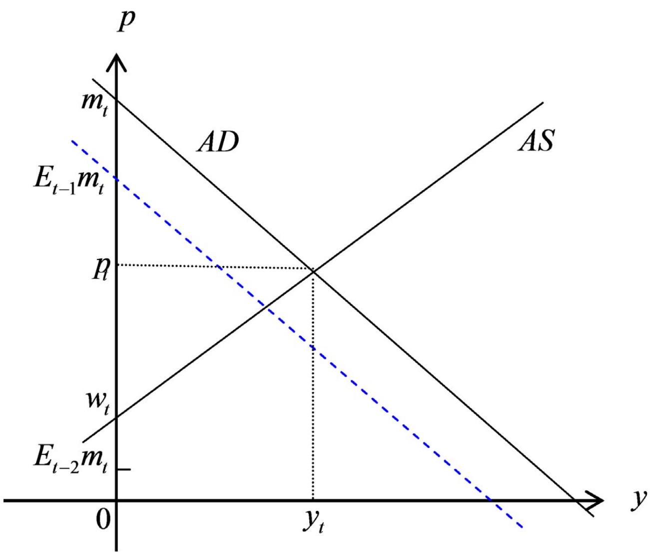

Figure 25.2: Equilibrium of the Fischer model in response to a positive monetary shock in period t (when also period $t-1$ featured a positive monetary shock).

of the solid downward-sloping AD curve is determined by the actual m_t. The stippled downward-sloping curve shows the position of the AD curve in case $m_t = E_{t-1}m_t$. The positive monetary shock implies that the equilibrium is located North-East of its position in the absence of the shock.[15]

The last equality in (25.40) reveals that there is a potential for real effects of monetary shocks to last two periods. To put it differently, it is always so that a significant fraction of the current wages has been set two periods in advance. Shocks in the current and the preceding period will thus matter for current output. But earlier shocks will not. After two periods, the price level has fully absorbed a monetary shock. A current shock, however, shows up partly in the current price and partly in the current output. In the next period there is still both an output effect and a price effect, but with more weight on the latter.

In the case illustrated by Fig. 25.2 there has been a monetary shock also in period $t-1$ in that $E_{t-1}m_t \neq E_{t-2}m_t$. Otherwise, the equilibrium corresponding to the stippled AD curve would be located on the vertical axis and thus have $y_t = 0$ and $w_t = p_t = m_t = E_{t-1}m_t = E_{t-2}m_t$ (remember, y is the output gap). Like in the benchmark model only the *unanticipated* part of money supply can affect output and make the output gap deviate from zero. This result is most clearly brought out by the last equality in (25.40).

[15] That the AS and AD curves are straight lines is due to the assumed linearity of the model. That these lines have slopes exactly equal to 1 and −1, respectively, is an artifact as explained in the preceeding footnote.

The overall conclusion is that in the Fischer model of asynchronous wage setting, real effects of monetary shocks are still not "truly" persistent; the effects last no longer than the contract period. Yet, if policy makers can act every period, the Fischer model allows *stabilization policy* to have a role. To see this, we introduce a demand shock by extending (25.33) into

$$y_t = m_t - p_t + u_t.$$

With m_t replaced by $m_t + u_t$,[16] the solution (25.40) takes the form

$$y_t = \frac{1}{3}\left[\frac{1}{2}(m_t + u_t - E_{t-1}(m_t + u_t)) + m_t + u_t - E_{t-2}(m_t + u_t)\right].$$

Suppose u_t follows a random walk, i.e.,

$$u_t = u_{t-1} + \varepsilon_t,$$

where ε_t is white noise with variance σ^2. Thus, $E_{t-1}u_t = u_{t-1}$ and $E_{t-2}u_t = u_{t-2}$. Consider two alternative monetary policies:

Policy 1. Suppose the central bank is known to follow a passive monetary policy:

$$m_t = \bar{m} \text{ for all } t, \tag{25.41}$$

where $\bar{m}$ is a constant. We get $E_{t-2}m_t = \bar{m} = E_{t-1}m_t$ and therefore

$$\begin{aligned} y_t &= \frac{1}{3}\left[\frac{1}{2}(u_t - E_{t-1}u_t) + u_t - E_{t-2}u_t\right] = \frac{1}{3}\left(\frac{1}{2}\varepsilon_t + u_{t-1} + \varepsilon_t - u_{t-2}\right) \\ &= \frac{1}{2}\varepsilon_t + \frac{1}{3}\varepsilon_{t-1}, \end{aligned}$$

so that the conditional variance of y_t as seen from the end of period $t-2$ is

$$\text{Var}_{t-2}(y_t) = (\frac{1}{4} + \frac{1}{9})\sigma^2.$$

Policy 2. Suppose the central bank is known to follow the counter-cyclical monetary policy:

$$m_t = -\varepsilon_{t-1}. \tag{25.42}$$

We get $E_{t-1}m_t = -\varepsilon_{t-1}$, $E_{t-2}m_t = E_{t-2}\left[-\varepsilon_{t-1}\right] = 0$, so that

$$\begin{aligned} y_t &= \frac{1}{3}\left[\frac{1}{2}(-\varepsilon_{t-1} + u_t + \varepsilon_{t-1} - u_{t-1}) - \varepsilon_{t-1} + u_t - u_{t-2}\right] \\ &= \frac{1}{3}\left(\frac{1}{2}\varepsilon_t - \varepsilon_{t-1} + u_{t-1} + \varepsilon_t - u_{t-2}\right) = \frac{1}{2}\varepsilon_t. \end{aligned} \tag{25.43}$$

[16] The natural interpretation of u_t is that it equals $-u_t^{LM}$, i.e., the negative of the velocity shock discussed in Section 25.41.

The conditional variance of y_t as seen from the end of period $t-2$ is now

$$\mathrm{Var}_{t-2}(y_t) = \frac{1}{4}\sigma^2,$$

which is smaller than under the passive monetary policy.

With the policy rule (25.42), m_t has become an *endogenous* variable in the system, partly offsetting the demand shock from the preceding period by moving in the opposite direction. One might ask: since shocks are not directly observable, how can monetary policy react to the shock ε_{t-1}? The answer is that this shock can be inferred from the observed y_{t-1}, as is seen from (25.43) if we replace t by $t-1$. Indeed, the monetary policy rule (25.42) can directly be written $m_t = -2y_{t-1}$.

25.5.2 A modified Fischer model

Here we modify the above model into a setup which is similar to Taylor's model of the subsequent section except in one key respect. The purpose is to highlight the fundamental source of persistent real effects of monetary shocks. We name this modified model *version II.* The model is:

$$y_t = m_t - p_t, \tag{25.44}$$

$$p_t = w_t = \frac{1}{2}(w_{t-1,t} + w_{t-2,t}), \tag{25.45}$$

$$w_{t-1,t} = E_{t-1}p_t + \alpha E_{t-1}y_t, \qquad w_{t-2,t} = E_{t-2}p_t + \alpha E_{t-2}y_t, \tag{25.46}$$

where $0 < \alpha < 1$. In a diagram corresponding to Fig. 25.2 the AD curve is thus unchanged, whereas the AS curve is now horizontal. The implicit (short-run) production function behind (25.45) is of the "fixed proportions type", $Y = \min(\xi K, \beta L)$, where ξ and β are constant technical coefficients. As long as the capital stock is not fully utilized, production is $Y = \beta L$.[17] Then, with price equal to a mark-up on marginal costs we have $P = \mu MC = \mu W/\beta$.[18] With $\mu = \beta$ (for simplicity) we get, in logs, $p = w$. The average wage w in period t is, as before, defined as $w_t = (\frac{1}{2})(w_{t-1,t} + w_{t-2,t})$. Equation (25.46) gives the wage, $w_{t-1,t}$, set (or negotiated) for period t by the group which at the end of period

[17] By empiricists this is sometimes considered a better approximation to reality under "normal circumstances" than the falling-marginal-product-of-labor approach of the first version of Fischer's model.

[18] So the real wage is $W/P = \beta/\mu$, a constant. Consequently, the real wage is here acyclical vis-a-vis demand shocks and not countercyclical as in the original Fischer model above and the benchmark model we started with. An acyclical real wage is actually a welcome feature of a model. The evidence does not point to countercyclical real wages but rather acyclical or slightly procyclical real wages (Stock and Watson, 1999).

$t-1$ sets wages for the next two periods, possibly at different levels. The wage set by this group for period $t+1$ will be $w_{t-1,t+1} = E_{t-1}p_{t+1} + \alpha E_{t-1}y_{t+1}$. Thus, the wage level a group sets for a given period depends on the expected price and the expected output (employment) in that period. The constant α is the elasticity of the group's target real wage w.r.t. employment; the assumption that this elasticity is positive reflects that workers' (unions') bargaining power increases with employment.

The employment-dependent target real wage adds to the realism of the model. We will name α the *local real wage flexibility*. The prefix "local" is important. Whereas the relative wages of the two groups will fluctuate depending on who is the first to act on new information, the global (or average) real wage level, $w - p$, remains constant, as indicated by (25.45).

By substituting (25.46) into (25.45) and taking expected values as seen from the end of period $t-2$, we see that $E_{t-2}y_t = 0$. By (25.44) then follows $E_{t-2}p_t = E_{t-2}m_t$. Agents know that after two periods any monetary shock is fully swallowed by the price level, leaving the output gap at its natural level, which is zero. Taking expected values as seen from the end of period $t-1$ in (25.45) then gives

$$E_{t-1}p_t = \frac{1}{2}(E_{t-1}p_t + \alpha E_{t-1}y_t + E_{t-2}m_t) = p_t, \tag{25.47}$$

by (25.45). Thus, $p_t = E_{t-1}p_t$. Indeed, p_t is cost determined and costs depend only on $w_{t-1,t}$ and $w_{t-2,t}$, both of which belong to the information available at the end of period $t-1$, as indicated by (25.46). To put it differently, at the end of period $t-1$, group A workers (or trade union) can in fact forecast p_t precisely, knowing their own decision, $w_{t-1,t}$, as well as group B's previous decision, $w_{t-2,t}$, and knowing the price formation function (25.45).

Taking expected values in (25.44) and substituting into (25.47) gives

$$\begin{aligned} p_t &= E_{t-1}p_t = \frac{\alpha}{1+\alpha}E_{t-1}m_t + \frac{1}{1+\alpha}E_{t-2}m_t \\ &= \frac{\alpha}{1+\alpha}(E_{t-1}m_t - E_{t-2}m_t) + E_{t-2}m_t. \end{aligned}$$

Finally, we substitute this into (25.44) to get

$$\begin{aligned} y_t &= m_t - \frac{\alpha}{1+\alpha}E_{t-1}m_t - \frac{1}{1+\alpha}E_{t-2}m_t \\ &= m_t - E_{t-1}m_t + \frac{1}{1+\alpha}(E_{t-1}m_t - E_{t-2}m_t). \end{aligned}$$

Again the effects on p_t and y_t of a monetary shock in period $t-1$ have the same sign and sum to $1\times$ (size of shock). A new aspect is, however, that now a shock in period t shows up fully in the current output, with no effect on the current price level. This is due to the horizontal AS curve the position of which is determined by w_t which is set in advance.[19] But w_t, hence also p_t, is affected by any shock in period $t-1$ and the more so, the higher is the local real wage flexibility, α. This means that the effect of such past shocks on y_t is *lower*, the higher is α. The mirror image of this is that the real effect of the shock is greater, the *higher* is local real wage *rigidity,* as measured by*, say,* $1/(1+\alpha)$.

We can rewrite the solution for output as

$$y_t = \frac{\alpha}{1+\alpha}(m_t - E_{t-1}m_t) + \frac{1}{1+\alpha}(m_t - E_{t-2}m_t).$$

Thus it still holds that only the unanticipated part of money supply affects output. The overall conclusion is that also in this version of the Fischer model of asynchronous wage setting, real effects of monetary shocks are not "truly" persistent.

25.6 Asynchronous wage setting with constant wage level for several periods: Taylor's model

As both versions of the Fisher model illustrated, asynchronous (staggered) wage setting as such is not enough to generate truly persistent effects of money on output. Indeed, the effects of monetary shocks did not last longer than the contract period. But the empirical evidence referred to above indicates persistent real effects. This is where Taylor's model comes in. Taylor (1979, 1980) incorporated the essential observation that wages (and prices) are often set at *constant values* for a fairly long time (say, N quarters of a year). When this is combined with the fact that the contract periods at the different local markets are staggered (overlapping), output persistence results, as we will now see.[20]

We present a simplified version of the Taylor model by letting $N = 2$. Then it becomes easy to compare it with Fischer's approach. Again one half of the labor force, group A, presets at the end of period $t-1$ its nominal wage level for period t and period $t+1$. The essential feature is that this level is *the same for the two periods*, i.e., $w_{t-1,t} = w_{t-1,t+1} = x_t$ (the contract wage of period t and period $t+1$). At the end of period $t+1$, group A resets the wage level for the next two periods, i.e., chooses x_{t+2}, and so on. The other half, group B, has at the end of

[19] Behind the story is the presumption that employment can adjust without an immediate wage increase. Thus, the existence of involuntary unemployment is presupposed.

[20] A pertinent question is, of course, *why* nominal contracts are fixed for so long and why indexing to the general price level is not used more often. Gray (1978) provides a study of this issue.

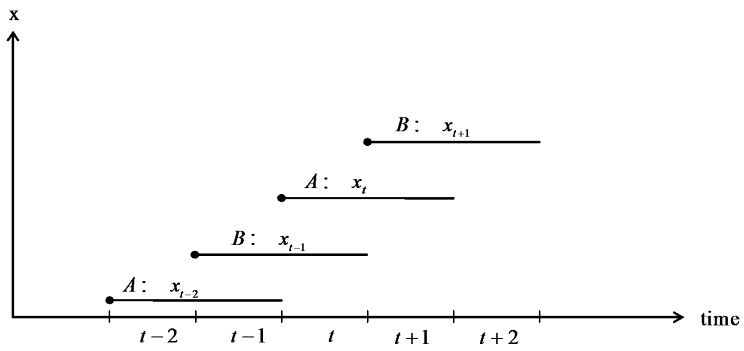

Figure 25.3: Configuration of wage setting in the Taylor model.

period $t-2$ preset its nominal wage level for period $t-1$ and period t at some level x_{t-1}, i.e., $w_{t-2,t-1} = w_{t-2,t} = x_{t-1}$. At the end of period t, group B resets its wage to some level, x_{t+1}, for the next two periods and so on. Fig. 25.3 illustrates. The wage set by a group for the subsequent two periods is set with the aim of achieving, in expected value, a certain target real wage. As in version II of the Fischer model this target real wage is assumed to be an increasing function of expected output (employment).

The Taylor model is:

$$y_t = m_t - p_t, \tag{25.48}$$

$$p_t = w_t \equiv \frac{1}{2}(w_{t-1,t} + w_{t-2,t}) = \frac{1}{2}(x_t + x_{t-1}), \tag{25.49}$$

$$x_t = \frac{1}{2}(E_{t-1}p_t + \alpha E_{t-1}y_t) + \frac{1}{2}(E_{t-1}p_{t+1} + \alpha E_{t-1}y_{t+1}), \tag{25.50}$$

where $0 < \alpha < 1$. Like in version II of the Fischer model, the implicit (short-run) production function behind the horizontal AS curve, (25.49), is of the "fixed proportions type". The average wage w in period t is, as before, defined as $w_t \equiv (\frac{1}{2})(w_{t-1,t} + w_{t-2,t})$, which in view of the constant wage during the contract period (i.e., $w_{t-2,t-1} = w_{t-2,t} = x_{t-1}$) can be expressed as the average of the current and the previous period's contract wages, $(x_t + x_{t-1})/2$. Equation (25.50) gives the wage, x_t, set (or negotiated) by the group which at the end of period $t-1$ sets its wage level to be in force in the next two periods. The two periods count equally much, since in a short-run perspective discounting is unimportant. Finally, the constant α also here indicates the elasticity of the local target real wage w.r.t. output (employment), thus reflecting the local real wage flexibility.

Before considering the solution to the model, we shall note a useful alternative interpretation of the wage setting behavior. We rewrite (25.50) as

$$\begin{aligned} x_t &= \frac{1}{2}(E_{t-1}p_t + E_{t-1}p_{t+1}) + \frac{\alpha}{2}(E_{t-1}y_t + E_{t-1}y_{t+1}) \\ &= \frac{1}{2}(p_t + E_{t-1}p_{t+1}) + \frac{\alpha}{2}(E_{t-1}y_t + E_{t-1}y_{t+1}), \end{aligned}$$

where we have used that $p_t = E_{t-1}p_t$. Indeed, p_t is cost determined and costs depend only on x_t and x_{t-1}, both of which belong to the information available at the end of period $t-1$, as indicated by (25.50). Inserting (25.49), both as it stands and in expected form with t replaced by $t+1$, gives

$$\begin{aligned} x_t &= \frac{1}{2}\left[\frac{1}{2}(x_t + x_{t-1}) + \frac{1}{2}(E_{t-1}x_{t+1} + x_t)\right] + \frac{\alpha}{2}(E_{t-1}y_t + E_{t-1}y_{t+1}) \\ &= \frac{1}{2}(x_{t-1} + E_{t-1}x_{t+1}) + \alpha(E_{t-1}y_t + E_{t-1}y_{t+1}). \end{aligned} \tag{25.51}$$

Thus, worker's wage setting behavior can be interpreted as if workers care about *relative* wages. Thus, the group of workers that sets wage at the end of period $t-1$ is concerned about x_{t-1} and $E_{t-1}x_{t+1}$, i.e., the wages paid to the other group in period t and period $t+1$. Since not only past expectations of current endogenous variables are present in the model, but also past expectations of *future* endogenous variables, finding the solution is somewhat more complicated than in the two previous models. The method we shall apply is called the *method of undetermined coefficients*.

Application of this method requires that we specify the stochastic process followed by the money supply. We assume money supply follows a *random walk*, i.e.,

$$m_t = m_{t-1} + \varepsilon_t, \tag{25.52}$$

where ε_t is white noise. Then, by the method of undetermined coefficients we find the reduced form of the model to be:[21]

$$x_t = (1-\lambda)m_{t-1} + \lambda x_{t-1}, \qquad \lambda \equiv \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} \in (0,1), \tag{25.53}$$

$$p_t = \frac{1}{2}\left[(1-\lambda)m_{t-1} + (1+\lambda)x_{t-1}\right], \tag{25.54}$$

$$y_t = \frac{1+\lambda}{2}(m_{t-1} - x_{t-1}) + \varepsilon_t. \tag{25.55}$$

The graph of λ as a function of the local-real-wage-flexibility parameter α is shown in Fig. 25.4. The negative dependency of λ on α invites seeing λ as an index of the degree of local real wage *rigidity*, which we may also call the *local real wage inertia*. The less relative prices respond to changes in demand, the greater is this "real rigidity" in the terminology of Ball and Romer (1990).

[21] The derivation is in Appendix D.

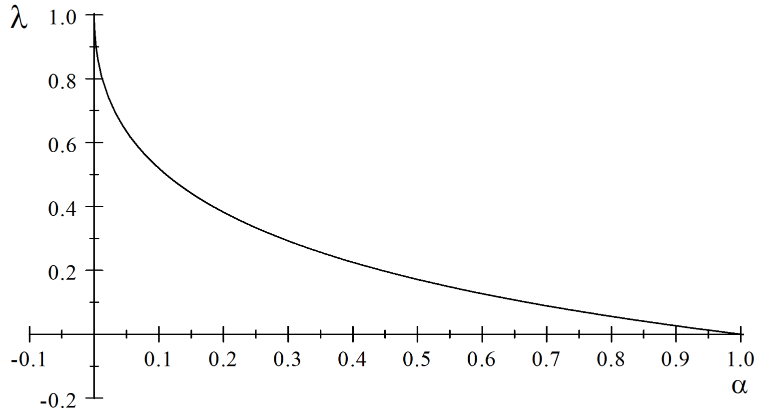

Figure 25.4: The graph of $\lambda = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}$.

The crucial feature is that now output has always a *backward* link. This is because output depends on demand, which depends negatively on the price level, which depends positively on the average wage level. But this wage level is formed as an average of the preset x_t and x_{t-1}, where x_t was set with a view of how x_{t-1} was set a period earlier. That is, not only do the expected circumstances in period t matter, but also the *expected circumstances in period* $t-1$ as seen from the end of period $t-2$.[22] And so on backward in time. The system never gets completely free from its previous history. Therefore, the effects of changes in the money supply last longer than the time during which each nominal wage is fixed. One way of getting the intuition behind the result is to recognize that workers care about relative wages, as shown by (25.51). Workers are therefore not only forward-looking, but also backward-looking.

To be more concrete, let us consider a positive permanent shock to the money supply. We assume the economy has been in steady state until period t. That is, we have $m_{t-i} = \bar{m}$ for $i = 1, 2, ...$. The corresponding steady-state values of x, p, and y are called $\bar{x}$, $\bar{p}$, and $\bar{y}$, respectively, and can be found in the following way. From (25.49) follows $\bar{p} = \bar{x}$. Then (25.50) gives $\bar{x} = \frac{1}{2}(2\bar{p} + 2\alpha\bar{y}) = \bar{x} + \alpha\bar{y}$. But this requires $\bar{y} = 0$. From (25.48) now follows $\bar{p} = \bar{x} = \bar{m}$.

Suppose an unanticipated positive permanent once-for-all money supply shock occurs in period t, i.e., $m_t = \bar{m} + \varepsilon \equiv \bar{m}' > \bar{m}$ and $m_{t+i} = \bar{m}'$ for $i = 1, 2, ...$ (no new shocks). The consequences of this shock are illustrated in Fig. 25.5, where the stipulated contract wages are indexed according to which group has set the

[22] This is in contrast to both versions of the Fischer model.

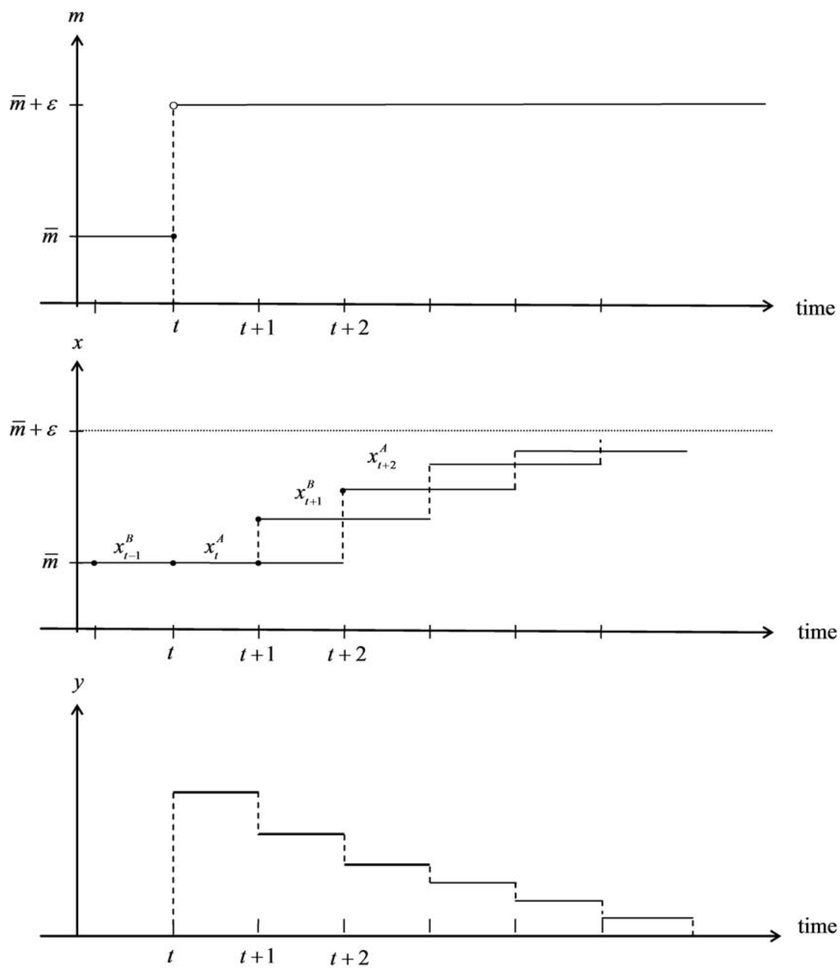

Figure 25.5: Adjustments to an unanticipated rise in the money supply.

wage. We have, from (25.53),

$$\begin{aligned}
x_t &= (1-\lambda)m_{t-1} + \lambda x_{t-1} = (1-\lambda)\bar{m} + \lambda\bar{m} = \bar{m}, \\
x_{t+1} &= (1-\lambda)m_t + \lambda x_t = (1-\lambda)(\bar{m}+\varepsilon) + \lambda\bar{m} = \bar{m} + (1-\lambda)\varepsilon, \\
x_{t+2} &= (1-\lambda)m_{t+1} + \lambda x_{t+1} = (1-\lambda)(\bar{m}+\varepsilon) + \lambda\bar{m} + \lambda(1-\lambda)\varepsilon \\
&= \bar{m} + (1-\lambda)(1+\lambda)\varepsilon, \\
x_{t+3} &= (1-\lambda)m_{t+2} + \lambda x_{t+2} = (1-\lambda)(\bar{m}+\varepsilon) + \lambda\bar{m} + \lambda(1-\lambda)(1+\lambda)\varepsilon, \\
&= \bar{m} + (1-\lambda)(1+\lambda+\lambda^2)\varepsilon, \\
&\quad \text{.......................................} \\
x_{t+i} &= \bar{m} + (1-\lambda)(1+\lambda+\lambda^2+...+\lambda^{i-1})\varepsilon = \bar{m} + (1-\lambda)\frac{1-\lambda^i}{1-\lambda}\varepsilon \\
&= \bar{m} + (1-\lambda^i)\varepsilon.
\end{aligned}$$

We see that

$$x_{t+i} \rightarrow \bar{m} + \varepsilon \text{ for } i \rightarrow \infty, \text{ since } 0 < \lambda < 1.$$

From (25.49) follows $p_t = \frac{1}{2}(x_t + x_{t-1}) = \bar{m}$ and, for $i = 1, 2, ...$,

$$\begin{aligned}
p_{t+i+1} &= \frac{1}{2}(x_{t+i+1} + x_{t+i}) = \frac{1}{2}\left[\bar{m} + (1-\lambda^{i+1})\varepsilon + \bar{m} + (1-\lambda^i)\varepsilon\right] \\
&= \bar{m} + \varepsilon - \lambda^i(1+\lambda)\varepsilon.
\end{aligned}$$

Thus,

$$p_{t+i+1} \rightarrow \bar{m} + \varepsilon \text{ for } i \rightarrow \infty.$$

Finally, by (25.55), $y_t = ((1+\lambda)/2)(m_{t-1} - x_{t-1}) + \varepsilon = \varepsilon$ and, for $i = 1, 2, ...$,

$$\begin{aligned}
y_{t+i+1} &= \frac{1+\lambda}{2}(m_{t+i} - x_{t+i}) + \varepsilon_{t+i+1} = \frac{1+\lambda}{2}\left[\bar{m} + \varepsilon - \bar{m} - (1-\lambda^i)\varepsilon\right] + 0 \\
&= \frac{1+\lambda}{2}\lambda^i\varepsilon,
\end{aligned}$$

so that

$$y_{t+i+1} \rightarrow 0 \text{ for } i \rightarrow \infty.$$

In period t, where the shock occurs, the price level is predetermined because prices are formed as a constant mark-up on predetermined marginal costs. Thus, the money shock has full effect on output in period t.[23] Thereafter the effect on output remains positive, but declining. This is because wages begin to adjust and so do prices. In the long run, if no new shocks occur, output returns to "natural".

[23] Like in version II of the Fischer model, this response of output without any current wage and price effect presupposes the existence of involuntary unemployment.

Note that the lower is α (local real wage flexibility), the closer is λ (local real wage inertia) to one and the slower is the return to the natural output level. Thus, *local real wage rigidity is important for persistence:* a higher λ results in higher persistence of the effect of monetary shocks on output.

The conclusion is that in addition to staggered wage contracts two factors contribute to significant persistence of real effects of a permanent money supply shock:

- nominal wage rigidity in the Taylor sense of local nominal wages fixed at the same level over several periods, and
- local real wage rigidity.

The first kind of rigidity results in nominal wages containing both a forward-looking and a backward-looking element (dependency of the past). The second kind of rigidity has to do with sluggishness in the response of *relative* prices (here the local target real wage). Higher local real wage rigidity, λ, which reflects lower sensitivity, α, of the target real wage to demand conditions, implies slower price adjustment. Therefore, the monetary easing in period t implies a higher real money supply for a longer time interval.[24]

Although in the above analysis only an unanticipated change in money supply was considered, it can be shown that in the Taylor model also an *anticipated* change in the money supply affects output. The intuition is that the contract wage x_t has both a backward-looking and a forward-looking component and begins its adjustment already at the time of the announcement of the future change in money supply. Indeed, the group of workers who set wages for the last period before the change in money supply is actually implemented and the first period of this implementation obviously takes the expected rise in the price level into account. This feeds back to the other group who sets wages one period earlier and so on backward in time to the period of the announcement, where the public becomes aware that a future change in money supply is on the way. As this change has not yet taken place, real money supply is gradually reduced in the periods between announcement and implementation. In the period where the announced policy change is implemented the price level is not yet fully adjusted (due to its backward link). Thus, real money supply is temporarily high and so is output. In the long run the price level adjusts fully, and output returns to the "natural" level (from above). In this manner the anticipated monetary easing in period t has a contractionary effect as long as it is expected, but not yet realized, and

[24] The importance of real rigidities in exacerbating sluggishness of price adjustment is highlighted by Ball and Romer (1990).

an expansionary effect when it is realized. But the expansionary effect gradually dies out as time proceeds.[25]

25.7 Conclusion

(preliminary, incomplete, messy)

In this chapter we have addressed macroeconomic themes where expectations in uncertain situations are important elements. This has been done within a simple New Keynesian framework:

- facing uncertainty due to stochastic disturbances to the money supply, the agents form model-consistent expectations;
- although wages and possibly also prices are sticky in the short run, they may change over time.

—

How well the RE hypothesis performs empirically is disputed. It is fair to say that the performance depends on the circumstances. Still, there is no generally accepted alternative workhorse hypothesis about expectations formation. When a model assumes rational expectations, we at least know that the emerging results will not depend on *systematic* expectation errors from the economic agents' side. In this sense the RE solution of a model may be considered a theoretically relevant benchmark case.

In some model setups (see next chapter) rational expectations lead to policy ineffectiveness. But as the above analysis has shown, when nominal rigidities in the form of staggered wage contracts are present, policy ineffectiveness is refuted. In the Fischer models as well as the Taylor model stabilization policy can be effective. Furthermore, in the Taylor model, but not in the Fischer models, money supply changes have persistent effects; shocks to money supply (or output demand in a more general model) have effects that last longer than the contract period. Outstanding wage contracts only gradually adjust to new information.

An analogous staggered pattern for prices prevails if price setters set prices at different times at different stages in the chain of production. Blanchard (1983) shows that this in itself can give rise to substantial price level inertia and non-neutrality of money even though prices change less often than wages. With combined wage and price staggering a *wage-price spiral* arises and the real effects of changes in money supply are reinforced, see Blanchard (1986), Edge (2002), and Rabanal and Rubio-Ramirez (2005).

[25] For details, see Andersen (1994).

A necessary, but not sufficient condition for weighty non-neutrality of money is the presence of some form of *rigidity in relative prices*, in particular real wages. Such "real price rigidities" can arise from wage bargaining, efficiency-wage mechanisms, insider-outsider relations, and search phenomena.

So far we have discussed models where wages and/or prices are set by different agents for *fixed periods of time* and revised only after fixed time intervals. Calvo (1982) developed a very popular model of price stickiness where each firm only has the option of changing price when it receives a random signal. Thus, in every period a constant fraction of the firms (randomly selected) make price decisions and these decisions are forward-looking, taking into account the expected path of the general price level. Inspired by this, Mankiw and Reis (2002) suggest a model where not prices, but information is sticky. In each period a constant fraction of firms (randomly selected) obtains new information about the state of the economy and computes a new path of optimal prices. Other firms continue to set prices based on old plans and old information. This structure also displays non-neutrality of money, and both price inertia and *inflation inertia* arise.

All these models focus on "time-dependent adjustment" in that plans can be revised only after a certain span of time or after arrival of a random signal. Another and probably more realistic approach is based on "state-dependent adjustment". Here adjustments follow so-called *Ss rules*. That is, adjustments only take place if the difference between actual price and a target price exceeds some threshold value (for an overview, see Blanchard 1990).

The fundamental issue why agents, and in particular firms, do not change prices more often is related to the existence of various kinds of price adjustment costs (menu costs etc.). We addressed this topic in Chapter **??**.

25.8 Literature notes

(incomplete)

The exposition in Section 25.3-6 draws heavily upon Blanchard (1990).

Critique of RE: Shiller in The New Palgrave, ...

Are there compelling alternatives? On "natural expectations", see

Learning: Ewans and Honkapohja ( ).

On the concept of information, see Rubinstein (1998) and Brunnermeier (2001). The latter book is a structured overview of advanced theory of information, asymmetric information, bubbles, crashes, and herding.

Fehr and Tyron (2005) survey research of relevance for the questions: When does a small amount of irrationality have large aggregate effects in the economy? And when does a small share of rational actors generate an aggregate outcome close to the prediction under the assumption that everybody are rational?

A systematic treatment of econometric methodology which sustains empirical modeling of economic time series is given in Hendry (1995). The econometric difficulties that arise because the stochastic properties of an an economy are not constant during economic development are discussed in Hendry and Mizon ( ).

25.9 Appendix

A. Conditional expectations and the law of iterated expectations

Generalities. We may think of the mathematical conditional expectation as a weighted sum of the possible values of the stochastic variable with weights equal to the corresponding conditional probabilities ("expectation" is used synonymously with "expected value").

Let Y and X be two *discrete* stochastic variables with joint probability function $h(y,x)$ and marginal probability functions $f(y)$ and $g(x)$, respectively. If the *conditional probability* function for Y given $X = x_0$ is denoted $j(y\,|x_0)$, we have $j(y\,|x_0) = h(y,x_0)/g(x_0)$, assuming $g(x_0) > 0$. The (mathematical) conditional expectation of Y given $X = x_0$, denoted $E(Y|X = x_0)$, is then

$$E(Y|X = x_0) = \sum_y y\frac{h(y,x_0)}{g(x_0)}. \qquad (25.56)$$

The summation is over all the possible values of y. This conditional expectation is a function of the given number x_0, perceived as a realization of X.

We may consider the conditional expectation itself as a new stochastic variable which is a function of the stochastic variable X. We may denote this new stochastic variable $E(Y|X)$. What is then its expectation? Generally, for a function of a discrete stochastic variable X, say $k(X)$, the expectation is

$$E(k(X)) = \sum_x k(x)g(x).$$

When we here let the conditional expectation $E(Y|X)$ play the role of $k(X)$ and sum over all $x \in S$ where S is the set of x-values for which $g(x) > 0$, we get

$$\begin{aligned} E(E(Y|X)) &= \sum_{x\in S} E(Y|X = x)g(x) = \sum_{x\in S}\left(\sum_y y\frac{h(y,x)}{g(x)}\right)g(x) \qquad \text{(by (25.56))} \\ &= \sum_{x\in S}\left(\sum_y yh(y,x)\right) = \sum_y y\left(\sum_{x\in S} h(y,x)\right) = \sum_y yf(y) = E(Y). \end{aligned}$$

This result is a manifestation of the *law of iterated expectations,* applied for instance in Section 25.2.2 and extensively applied in the next chapter. The law says that the unconditional expectation of the conditional expectation of Y is given by the unconditional expectation of Y.

Now consider the case where Y and X are *continuous* stochastic variables with joint probability *density* function $h(y,x)$ and marginal density functions $f(y)$ and $g(x)$, respectively. The *conditional density* function for Y given $X = x_0$, denoted $j(y\,|x_0)$, is defined by $j(y\,|x_0) = h(y,x_0)/g(x_0)$, if $g(x_0) > 0$, and is not defined otherwise.[26] With the range of Y being, for instance, $(-\infty, \infty)$, the conditional expectation of Y given $X = x_0$, where $g(x_0) > 0$, is

$$E(Y|X = x_0) = \int_{-\infty}^{\infty} y \frac{h(y,x_0)}{g(x_0)} dy. \tag{25.57}$$

Again, we may view the conditional expectation itself as a stochastic variable, writing it as $E(Y|X)$. Generally, for a function of a continuous stochastic variable X, say $k(X)$, the expected value is

$$E(k(X)) = \int_S k(x)g(x)dx,$$

where S is the set of x-values for which $g(x) > 0$. When we here let the conditional expectation $E(Y|X)$ play the role of $k(X)$, we get

$$\begin{aligned} E(E(Y|X)) &= \int_S E(Y|X = x)g(x)dx = \int_S \left(\int_{-\infty}^{\infty} y \frac{h(y,x)}{g(x)} dy \right) g(x)dx \\ &= \int_{-\infty}^{\infty} y \left(\int_S h(y,x)dx \right) dy = \int_{-\infty}^{\infty} yf(y)dy = E(Y). \end{aligned} \tag{25.58}$$

This shows us the *law of iterated expectations* in action for a continuous stochastic variable: the unconditional expectation of the conditional expectation of Y is given by the unconditional expectation of Y.

EXAMPLE Let the two stochastic variables, X and Y, follow a two-dimensional normal distribution. Then, from mathematical statistics we know that the conditional expectation of Y given X satisfies

$$E(Y|X) = E(Y) + \frac{\text{Cov}(Y,X)}{\text{Var}(X)}(X - E(X)).$$

Taking expectations on both sides gives

$$E(E(Y|X)) = E(Y) + \frac{\text{Cov}(Y,X)}{\text{Var}(X)}(E(X) - E(X)) = E(Y). \qquad \square$$

[26] In spite of the event $X = x_0$ generally having zero probability when X is a continuous stochastic variable (recall that this does not contradict that $g(x_0) > 0$), performing the appropriate integrations tells us that this definition results in properties corresponding to our intuitive notions of conditional probability and conditional expectation.

As an extension, let the conditioning stochastic variable X be a vector, $X = (X_1, X_2)$, in $\mathbb{R}^2$. Let Y, X_1, and X_2 have joint density function $h(y, x_1, x_2)$, and Y and X_1 have joint density function $m(y, x_1)$, whereas X_1 and X_2 have joint density function $g(x_1, x_2)$. In addition, let X_1 have marginal density function $n(x_1)$.

The conditional expectation of Y given $X_1 = x_1$ and $X_2 = x_2$, where $g(x_1, x_2) > 0$, is

$$E(Y|X_1 = x_1, X_2 = x_2) = \int_{-\infty}^{\infty} y \frac{h(y, x_1, x_2)}{g(x_1, x_2)} dy. \tag{25.59}$$

Let $E(Y|X_1 = x_1, X_2)$ denote the conditional expectation of Y given $X_1 = x_1$, but *before* the realization of X_2. With S denoting the set of x_2-values for which $g(x_1, x_2) > 0$, we consider this conditional expectation as a function of the stochastic variable X_2 and calculate its expected value:

$$\begin{aligned} E(E(Y|X_1 &= x_1, X_2)) = \int_S \left(\int_{-\infty}^{\infty} y \frac{h(y, x_1, x_2)}{g(x_1, x_2)} dy \right) \frac{g(x_1, x_2)}{n(x_1)} dx_2 \\ &= \int_S \left(\int_{-\infty}^{\infty} y \frac{h(y, x_1, x_2)}{n(x_1)} dy \right) dx_2 = \int_{-\infty}^{\infty} y \left(\frac{\int_S h(y, x_1, x_2) dx_2}{n(x_1)} \right) dy \\ &= \int_{-\infty}^{\infty} y \frac{m(y, x_1)}{n(x_1)} dy = E(Y|X_1 = x_1). \end{aligned}$$

In brief,

$$E(E(Y|X_1 = x_1, X_2)) = E(Y|X_1 = x_1), \tag{25.60}$$

which is the law of iterated expectations for this case.

Information sets and Wenn diagrams: a warning. In a *dynamic* context we typically consider periods, say, $1, 2, \ldots, t, \ldots$, and interpret $x_1, x_2, \ldots, x_t, \ldots$, as successive realizations of $X_1, X_2, \ldots, X_t, \ldots$, whereas Y could refer to period $t+1$. If $t = 2$, the expression (25.60) then illustrates the law of iterated expectations in a dynamic perspective.

In line with this, the main text expressed the conditioning explicitly in terms of dated *information sets*, $\ldots, I_{t-1}$, I_t, $I_{t+1}, \ldots$. This is convenient in a dynamic context. These dated information sets satisfy the *inclusion relationship* $I_{t-1} \subseteq I_t$, and the law of iterated expectations takes the form

$$E(E(Y_{t+1}|I_t)|I_{t-1}) = E(Y_{t+1}|I_{t-1}). \tag{25.61}$$

In words: the expectation, conditional of the information up to period $t-1$, of the conditional expectation of Y_{t+1}, given the information up to period t, equals the expectation of Y_{t+1}, conditional of the information up to period $t-1$. So the expectation today of the expectation tomorrow, when more may be known, of a variable the day after tomorrow is the same as the expectation today of the variable the day after tomorrow.[27]

What is the relationship between these information sets and *Wenn diagrams* as used for elementary instruction in set theory? Let again the situation described in (25.60) be our example. Let the event A be that $(X_1, X_2) = (x_1, \text{unknown})$, whereas the event B is that $(X_1, X_2) = (x_1, x_2)$. So B is a proper subset of A, $B \subset A$. Before the realization of X_2 is known, $E(Y\,|B)$ is a stochastic variable. Then (25.60) can be written:

$$E(E(Y\,|B)\,|A) = E(Y\,|A)\,. \tag{25.62}$$

So, in terms of the subsets A and B of the outcome space for the conditioning variables, the law of iterated expectations says that the expectation, conditional on A, of the expectation of Y, conditional on B, is the same as the expectation, conditional on A, of Y.

The analogy with (25.61) with $t = 2$ is that given A, only $X_1 = x_1$ is known, which corresponds to the information set I_{t-1}, whereas given B, both $X_1 = x_1$ and $X_2 = x_2$ are known, corresponding to the information set I_t. But now we face a perplexing feature. The inclusion relationship between B and A in (25.62) is *opposite* of that between the "corresponding" information sets, I_t and I_{t-1}, in (25.61), that is, $B \subseteq A$ whereas $I_t \supseteq I_{t-1}$. Although not really visible in the notation to the right of the separator "|", the inclusion relationship between subsets of the outcome space for the conditioning variables is thus the opposite of that between the corresponding information sets.

The resolution of the seeming paradox is as follows. If what appears to the right of "|" is subsets, like A and B, of the outcome space for the conditioning variables, here X, then, in spite of $B \subset A$ looking as if B were *less* informative than A, the concept of *information* implies the opposite. Indeed, B is *more* informative than A in the sense that B rules out a *larger* set of ex ante outcome possibilities for X_1 and X_2 than A. Actually, B rules out the *complement* of B, B^c, and it is B^c that is larger than the complement of A, A^c, hence making B more informative than A. In B more of the uncertainty has been eliminated. *This* is in line with $I_t \supseteq I_{t-1}$.

What we learn from this is that the standard notation is tricky. It does not clearly discriminate between the case where to the right of the separator "|" appears subsets of the outcome space for the conditioning variables and the case where the corresponding information sets appear.

[27] In particular in the subsequent chapters does this principle play a key role.

B. Properties of the model-consistent forecast

As in the text of Section 25.2.2, let e_t denote the model-consistent forecast error $Y_t - E(Y_t|I_{t-1})$. Then, if S_{t-1} is a subset of the information set I_{t-1},

$$\begin{aligned} E(e_t\,|S_{t-1}) &= E(Y_t - E(Y_t\,|I_{t-1})\,|S_{t-1}) = E(Y_t\,|S_{t-1}) - E(E(Y_t\,|I_{t-1})\,|S_{t-1}) \\ &= E(Y_t\,|S_{t-1}) - E(Y_t\,|S_{t-1}) = 0, \end{aligned} \tag{25.63}$$

where we have used that $E(E(Y_t\,|I_{t-1})\,|S_{t-1}) = E(Y_t\,|S_{t-1})$, by the law of iterated expectations, cf. Appendix A. With $S_{t-1} = I_{t-1}$ we have, as a special case,

$$\begin{aligned} E(e_t\,|I_{t-1}) &= 0, \quad \text{as well as} \qquad (25.64) \\ E(e_t) &= E(Y_t - E(Y_t\,|I_{t-1})) = E(Y_t) - E(E(Y_t\,|I_{t-1})) = 0, \end{aligned}$$

in view of the law of iterated expectations. This proves property (a) in Section 25.2.2.

As to property (b), for $i = 1, 2, ...,$ let s_{t-i} be an arbitrary variable value belonging to the information I_{t-i}. Then, $E(e_t s_{t-i}\,|I_{t-i}) = s_{t-i}E(e_t\,|I_{t-i}) = 0$, by (25.63) with $S_{t-1} = I_{t-i}$ (since I_{t-i} is contained in I_{t-1}). Thus, by the law of iterated expectations,

$$E(e_t s_{t-i}) = E\left(E(e_t s_{t-i}\,|I_{t-i})\right) = E(0) = 0 \quad \text{for } i = 1, 2, ... \tag{25.65}$$

This result is known as the *orthogonality property* of model-consistent expectations (two stochastic variables Z and V are said to be *orthogonal* if $E(ZV) = 0$). From the general formula for the (unconditional) covariance follows

$$\mathrm{Cov}(e_t s_{t-i}) = E(e_t s_{t-i}) - E(e_t)E(s_{t-i}) = 0 - 0 = 0, \quad \text{for } i = 1, 2, ...,$$

by (25.64) and (25.65). In particular, with $s_{t-i} = e_{t-i}$, we get $\mathrm{Cov}(e_t e_{t-i}) = 0$. This proves that model-consistent forecast errors exhibit *lack of serial correlation.*

C. The log-linear specification

In many macroeconomic models with rational expectations the equations are specified as log-linear, that is, as being linear in the logarithms of the variables. If Y, X, and Z are the original positive stochastic variables, defining $y = \ln Y$, $x = \ln X$, and $z = \ln Z$, a log-linear relationship between Y, X, and Z is a relation of the form

$$y = \alpha + \beta x + \gamma z, \tag{25.66}$$

where α, β, and γ are constants. The motivation for assuming log-linearity can be:

(a) Linearity is convenient because of the simple rule for the expected value of a sum: $E(\alpha + \beta x + \gamma z) = \alpha + \beta E(x) + \gamma E(z)$, where E is the expectation operator. Indeed, for a non-linear function, $f(x, z)$, we generally have $E(f(x, z)) \neq f(E(x), E(z))$.

(b) Linearity in logs may often seem a more realistic assumption than linearity in anything else.

(c) In time series models a logarithmic transformation of the variables followed by formation of first differences can be the road to eliminating a trend in the mean and variance.

As to point (b) we state the following:

CLAIM To assume linearity in logs is equivalent to assuming constant elasticities.

Proof Let the positive variables Y, X and Z be related by $Y = F(X, Z)$, where F is a continuous function with continuous partial derivatives. Taking the differential on both sides of $\ln Y = \ln F(X, Z)$, we get

$$\begin{aligned} d\ln Y &= \frac{1}{F(X,Z)}\frac{\partial F}{\partial X}dX + \frac{1}{F(X,Z)}\frac{\partial F}{\partial Z}dZ \quad (25.67) \\ &= \frac{X}{Y}\frac{\partial Y}{\partial X}\frac{dX}{X} + \frac{Z}{Y}\frac{\partial Y}{\partial Z}\frac{dZ}{Z} = \eta_{YX}\frac{dX}{X} + \eta_{YZ}\frac{dZ}{Z} = \eta_{YX}d\ln X + \eta_{YZ}d\ln Z, \end{aligned}$$

where η_{YX} and η_{YZ} are the partial elasticities of Y w.r.t. X and Z, respectively. Thus, defining $y = \ln Y$, $x = \ln X$, and $z = \ln Z$, gives

$$dy = \eta_{YX}dx + \eta_{YZ}dz. \quad (25.68)$$

Assuming constant elasticities amounts to putting $\eta_{YX} = \beta$ and $\eta_{YZ} = \gamma$, where β and γ are constants. Then we can write (25.68) as $dy = \beta dx + \gamma dz$. By integration, we get (25.66) where α is now an arbitrary integration constant. Hereby we have shown that constant elasticities imply a log-linear relationship between the variables.

Now, let us instead start by assuming the log-linear relationship (25.66). Then,

$$\frac{\partial y}{\partial x} = \beta, \frac{\partial y}{\partial z} = \gamma. \quad (25.69)$$

But (25.66), together with the definitions of y, x and z, implies that

$$Y = e^{\alpha + \beta x + \gamma z} = e^{\alpha + \beta \ln X + \gamma \ln Z},$$

from which follows that

$$\frac{\partial Y}{\partial X} = Y\beta\frac{1}{X} \text{ so that } \eta_{YX} \equiv \frac{X}{Y}\frac{\partial Y}{\partial X} = \beta,$$

and
$$\frac{\partial Y}{\partial Z} = Y\gamma\frac{1}{Z} \text{ so that } \eta_{YZ} \equiv \frac{Z}{Y}\frac{\partial Y}{\partial Z} = \gamma.$$
That is, the partial elasticities are constant. □

So, when the variables are in logs, then the coefficients in the linear expressions are the elasticities. Note, however, that the interest rate is normally an exception. It is often regarded as more realistic to let the interest rate itself and not its logarithm enter linearly. Then the associated coefficient indicates the *semi-elasticity* with respect to the interest rate.

D. Solving the Taylor model

With money supply following a random walk, the Taylor model can be reduced to a stochastic difference equation in only one endogenous variable, the contract wage x_t. To accomplish that, we eliminate the other endogenous variables, y and p, and the expected values of these. In (25.48), as it reads and shifted one period ahead, we substitute (25.49) and take expected values on both sides, using the random walk hypothesis, (25.52),

$$\begin{aligned} E_{t-1}y_t &= m_{t-1} - \frac{1}{2}x_t - \frac{1}{2}x_{t-1}, \quad \text{and} \\ E_{t-1}y_{t+1} &= m_{t-1} - \frac{1}{2}E_{t-1}x_{t+1} - \frac{1}{2}x_t, \end{aligned}$$

where we have applied that $E_{t-1}x_t = x_t$ (remember, x_t is set at the end of period $t-1$ and thus belongs to the information I_{t-1}). Substituting these expressions into (25.51), we get

$$\begin{aligned} x_t &= \frac{1}{2}(x_{t-1} + E_{t-1}x_{t+1}) + \alpha\left[2m_{t-1} - x_t - \frac{1}{2}x_{t-1} - \frac{1}{2}E_{t-1}x_{t+1}\right] \\ &= \frac{1}{2}(1-\alpha)(x_{t-1} + E_{t-1}x_{t+1}) + \alpha\left[2m_{t-1} - x_t\right]. \end{aligned} \tag{25.70}$$

Reordering yields

$$\begin{aligned} x_t &= \frac{1-\alpha}{2(1+\alpha)}(x_{t-1} + E_{t-1}x_{t+1}) + \frac{2\alpha}{1+\alpha}m_{t-1} \\ &\equiv \frac{\gamma}{2}(x_{t-1} + E_{t-1}x_{t+1}) + (1-\gamma)m_{t-1}, \text{ with } \gamma \equiv \frac{1-\alpha}{1+\alpha}, \end{aligned} \tag{25.71}$$

where $0 < \gamma < 1$.

This is a linear second-order stochastic difference equation. Applying the *method of undetermined coefficients,* an informed conjecture is that any (non-explosive) solution is a linear function (with constant coefficients) of the lagged values of x and m. Hence, we guess the solution takes the form:

$$x_t = \lambda x_{t-1} + \beta m_{t-1}, \tag{25.72}$$

where the coefficients λ and β are to be determined. We forward this expression one period and take period $t-1$ expectations (again using that $E_{t-1}x_t = x_t$ and $E_{t-1}m_t = m_{t-1}$):

$$E_{t-1}x_{t+1} = \lambda E_{t-1}x_t + \beta E_{t-1}m_t = \lambda x_t + \beta m_{t-1}.$$

Then, we substitute into (25.71) to get

$$x_t = \frac{\gamma}{2}(x_{t-1} + \lambda x_t + \beta m_{t-1}) + (1-\gamma)m_{t-1}.$$

By ordering,

$$x_t = \frac{\gamma}{2-\gamma\lambda}x_{t-1} + \frac{\gamma\beta + 2(1-\gamma)}{2-\gamma\lambda}m_{t-1}.$$

Comparing with (25.72), we see that for our conjecture to be correct, we must have

$$\frac{\gamma}{2-\gamma\lambda} = \lambda \quad \text{and} \tag{25.73}$$

$$\frac{\gamma\beta + 2(1-\gamma)}{2-\gamma\lambda} = \beta. \tag{25.74}$$

Equation (25.73) implies the quadratic equation

$$\gamma\lambda^2 - 2\lambda + \gamma = 0, \tag{25.75}$$

which has the solution

$$\lambda = \frac{1 \overset{(+)}{-} \sqrt{1-\gamma^2}}{\gamma} = \frac{1+\alpha-2\sqrt{\alpha}}{1-\alpha} = \frac{(1-\sqrt{\alpha})^2}{1-(\sqrt{\alpha})^2} = \frac{(1-\sqrt{\alpha})^2}{(1+\sqrt{\alpha})(1-\sqrt{\alpha})} = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}.$$

where we have used that $\gamma \equiv (1-\alpha)/(1+\alpha) \in (0,1)$ so that $1-\gamma^2 = 4\alpha/(1+\alpha)^2$. We have excluded the other root, which is above 1 and therefore implies an explosive solution. Next we find the coefficient β. From (25.74) we have $\gamma\beta + 2(1-\gamma) = (2-\gamma\lambda)\beta = \gamma\beta/\lambda$, by (25.73). Ordering yields

$$\begin{aligned} \gamma\lambda\beta + 2\lambda - 2\gamma\lambda &= \gamma\beta \Rightarrow \\ \gamma\lambda\beta + \gamma\lambda^2 + \gamma - 2\gamma\lambda &= \gamma\beta \quad \text{(by (25.75))} \end{aligned}$$

We now divide through by γ and get $\lambda\beta + \lambda^2 + 1 - 2\lambda = \beta \Rightarrow$

$$\begin{aligned}(1-\lambda)\beta &= (1-\lambda)^2 \Rightarrow \\ \beta &= 1-\lambda.\end{aligned}$$

With these values of λ and β, by way of construction, (25.72) *is* a solution of (25.71), thus confirming (25.53) in Section 19.5. Then, (25.54) and (25.55) follow by (25.49) and (25.48), respectively. If wage setters believe the economy moves according to (25.53), (25.54), and (25.55), the aggregate effects of their wage setting behavior will be such that their beliefs are confirmed.

25.10 Exercises

25.1. Suppose that $Y_t = X_t + e_t$, where $\{X_t\}$ is a random walk and e_t is white noise.

a) What is the rational expectation of Y_t conditional on all relevant information up to and including period $t-1$?

b) Compare with the subjective expectation of Y_t based on the static expectations formula (the adaptive expectations formula with adjustment speed equal to one).

c) Compare the rational expectation of X_t with the static expectation as seen from period $t-1$

25.2. Consider a simple Keynesian model of a closed economy with constant wages and prices (behind the scene), abundant capacity, and output determined by demand:

$$\begin{aligned}Y_t &= D_t = C_t + \bar{I} + G_t, && (1)\\ C_t &= \alpha + \beta Y^e_{t-1,t}, \qquad \alpha > 0,\ 0 < \beta < 1, && (2)\\ G_t &= (1-\rho)\bar{G} + \rho G_{t-1} + \varepsilon_t, \qquad \bar{G} > 0,\ 0 < \rho < 1, && (3)\end{aligned}$$

where the endogenous variables are Y_t = output (= income), D_t = aggregate demand, C_t = consumption, and $Y^e_{t-1,t}$ = expected output (income) in period t as seen from period $t-1$, while G_t, which stands for government spending on goods and services, is considered exogenous as is ε_t, which is white noise. Finally, investment, $\bar{I}$, and the parameters α, β, ρ, and $\bar{G}$ are given positive constants.

Suppose expectations are "static" in the sense that expected income in period t equals actual income in the previous period.

a) Solve for Y_t.

b) Find the income multiplier (partial derivative of Y_t) with respect to a change in G_{t-1} and ε_t, respectively.

Suppose instead that expectations are rational.

c) Explain what this means.

d) Solve for Y_t.

e) Find the income multiplier with respect to a change in G_{t-1} and ε_t, respectively.

f) Compare the result under e) with that under b). Comment.