Chapter 17

Inflation and capital accumulation: The Sidrauski model

In this chapter we consider a competitive economy from the perspective of key questions in monetary macroeconomics. The questions studied are: (a) How do the size and the growth rate of the money supply affect resource allocation, price level, inflation rate, and welfare in the economy? (b) Given the preferences of a representative household, what rule should govern the quantity of money? (c) Is hyperinflation always a consequence of excessive growth in the money supply or can it be generated by self-fulfilling expectations? (d) Can deflation be generated by self-fulfilling expectations?

One approach to these questions is exemplified by a contribution by the Argentine economist Miguel Sidrauski (1939-1968), graduated from University of Chicago. Shortly before his tragic death at the age of twenty eight he published a famous paper based on a monetary Ramsey model (Sidrauski 1967a). Prices and wages are assumed fully flexible and markets are competitive. The model is thus an example of a *neoclassical* monetary model. Whatever its worth as a theory of the actual macroeconomic functioning of money, the model exposes how changes in the money supply would operate in a world of full capacity utilization and fully flexible prices of goods and labor.

Among other things, the Sidrauski model leads to Milton Friedman's famous and controversial zero interest rate rule (Friedman 1969). This rule recommends a deflationary monetary policy such that the opportunity cost of holding cash, the nominal interest rate, becomes zero.[1]

[1] Milton Friedman (1912-2006), who spent most of his academic life at the University of Chicago, was the leading figure in the school of thought called *monetarism*.

17.1 The agents

There are three types of agents in the model, households, firms, and a consolidated government-central bank. The model ignores the private banking sector. So "money" in the model means *base money*. For simplicity, technology is constant and there are no capital adjustment costs. Time is continuous.

The households

Following the Ramsey setup we consider a fixed number of households − or rather dynastic families − with infinite time horizon. The families are identical. Every family has L_t members at time t, and L_t grows at the constant rate $n \geq 0$. Every family member inelastically supplies one unit of labor per time unit. We normalize the number of households to 1 such that L_t also measures the aggregate labor supply. The family is fortunate to be equipped with perfect foresight.

Let the family's total consumption be denoted by C_t and its total desired nominal money holdings by M_t. The corresponding per capita quantities are

$$c_t \equiv \frac{C_t}{L_t}, \quad \text{and} \quad m_t \equiv \frac{M_t}{P_t L_t},$$

where P_t is the general price level measured in current money, i.e., the GDP deflator. The instantaneous utility is given by the utility function

$$u(c_t, m_t), \qquad u_c > 0, u_m > 0, u_{cc} < 0, u_{mm} < 0. \tag{17.1}$$

That is, current utility at time t depends positively on current consumption per head, c_t, and real money holdings per head, m_t.[2] People wish to possess a certain liquidity to lessen transaction efforts (these efforts or "transaction costs" are not explicit in the model, but should be understood in some non-pecuniary sense). Notice that the liquidity service of money depends on the purchasing power of money, that is, the argument in the utility function is $(M_t/L_t)/P_t$, not M_t/L_t. Postulating that the liquidity service of money contributes directly to utility instead of merely being a means to reduce transaction effort, is not entirely satisfactory, of course. It is a short-cut which is analytically convenient.

The family wants to maximize discounted utility, $\int_0^\infty u(c_t, m_t) L_t e^{-\rho t} dt$, where ρ is the rate of time preference. For simplicity, we concentrate on a special case of the instantaneous utility function, namely

$$u(c_t, m_t) = \frac{c_t^{1-\theta} - 1}{1-\theta} + \alpha \frac{m_t^{1-\varepsilon} - 1}{1-\varepsilon},$$

[2] Other than the stated properties of u, one might, for the general case of non-separability of u, want to add a requirement that u is concave (and thereby $u_{cc}u_{mm} - u_{cm}^2 \geq 0$). We shall be concerned with the case where u is additively separable and then concavity of u is automatically satisfied.

where θ, ε, and α are given positive parameters. That is, the utility contributions from consumption and money holdings, respectively, enter in an additive separable way in the form of two CRRA functions with elasticity of marginal utility of θ and ε, respectively.[3] The greater θ and ε, respectively, the greater the desire to smooth consumption and money holdings, respectively, over time. The parameter α expresses the weight attached to money holdings relative to consumption. Hence, the objective function of the family, as seen from time 0, can be written

$$U_0 = \int_0^\infty \left(\frac{c_t^{1-\theta} - 1}{1-\theta} + \alpha \frac{m_t^{1-\varepsilon} - 1}{1-\varepsilon} \right) e^{-(\rho-n)t} dt, \qquad (17.2)$$

where an unimportant positive factor, L_0, has been eliminated. To ease convergence of the utility integral for $t \to \infty$, we presume that the *effective* discount rate $\bar{\rho} \equiv \rho - n$ is positive.

Let

$$\begin{aligned} A_t &\equiv \text{ real financial wealth,} \\ V_t &\equiv A_t - \frac{M_t}{P_t} \equiv \text{ real non-monetary financial wealth.} \end{aligned} \qquad (17.3)$$

The non-monetary financial wealth, sometimes called interest-bearing wealth, may consist of capital goods, bonds, and shares of stock. The increase per time unit in the family's financial wealth is

$$\dot{A}_t = r_t(A_t - M_t/P_t) - \pi_t M_t/P_t + w_t L_t + X_t - L_t c_t, \quad A_0 \text{ given.} \qquad (17.4)$$

Here, r_t is the real interest rate, $\pi_t \equiv \dot{P}_t/P_t$ is the rate of inflation, w_t is the real wage, and X_t is lump-sum transfers from the government. The transfers are financed by money issue (see below), and there is no taxation. Considering (17.4) as an ex post accounting relationship, r_t will be interpreted as the ex post realized real interest rate $i_t - \pi_t$, where i_t is the short-term nominal interest rate. From an ex ante point of view the economically relevant interest rate is the *expected* real interest rate which is the observable nominal interest rate, i_t, minus the *expected* inflation rate, π_t^e. But as uncertainty is ignored, expectations are assumed always to be validated ex post so that $\pi_t^e = \pi_t$ and we simply have $r_t = i_t - \pi_t$ ex ante as well as ex post.

The absence of uncertainty implies that all interest-bearing assets earn the same rate of return. Otherwise nobody will hold the asset with the lower return.

[3] To ease graphical illustration, cf. Fig. 17.2, we have written the two CRRA functions in "normalized form" by subtracting the constants $1/(1-\theta)$ and $1/(1-\varepsilon)$, respectively. As usual, the CRRA formulas should be interpreted as $\ln c_t$ and/or $\ln m_t$, if $\theta = 1$ and/or $\varepsilon = 1$, respectively.

Money, however, is a different kind of asset. Owing to the liquidity services it provides, it will generally bear a lower rate of return than other financial assets. Indeed, the *rate* of return on holding (base) money is, by definition, $[d(1/P_t)/dt]/(1/P_t) = -P_t^{-2}\dot{P}_t/(1/P_t) = -\dot{P}_t/P_t \equiv -\pi_t$, which is negative, when inflation is positive.

Now, the dynamic budget identity (17.4) says that saving (i.e., the increase per time unit in financial wealth) equals income minus consumption. The income is composed of capital income, labor income, $w_t L_t$, and transfers, X_t. Capital income is the return on financial wealth and this return consists of two terms, first, the return on interest-bearing financial wealth, $r_t(A_t - M_t/P_t)$, second, the "return" on the liquid part of financial wealth, $-\pi_t M_t/P_t$, which is usually negative. Stated differently, $\pi_t M_t/P_t$ is the "capital loss" by holding part of the wealth in liquids and thereby being exposed to depreciation in the real value of this part of wealth.[4]

By inserting $M_t/P_t \equiv L_t m_t$, (17.4) takes the form

$$\dot{A}_t = r_t(A_t - L_t m_t) - \pi_t L_t m_t + w_t L_t + X_t - L_t c_t, \qquad A_0 \text{ given.} \tag{17.5}$$

It is not until this dynamic book-keeping is supplemented by a requirement of solvency, that we have a budget *constraint* restricting the intertemporal consumption path. The solvency requirement is the No-Ponzi-Game condition

$$\lim_{t\to\infty} A_t e^{-\int_0^t r_s ds} \geq 0. \tag{NPG}$$

This condition implies that far out in the future, the household (or "family") can still have a negative net financial position ($A < 0$), the absolute value of which is even growing over time although, at most, at a rate *less* than the interest rate.

Note that in contrast to previous chapters, we here have *two* assets, an interest-bearing asset and money. This might raise the question why the relevant solvency requirement takes exactly this form in the present model where we have *two* assets, money and a non-monetary asset, so that $A_t \equiv M_t/P_t + V_t$. The answer is given in Appendix A.

The optimization problem of the household is: choose a plan $(c_t, m_t)_{t=0}^{\infty}$ such that a maximum of U_0 is achieved subject to the constraints (17.5) and (NPG). One could proceed by using the Maximum Principle directly on this problem with two control variables, c_t and m_t, and one state variable, A_t. We will use the alternative procedure where the problem is first transformed into per capita terms.

[4]Another way of understanding (17.4) is as follows: by differentiating w.r.t. t in (17.3) we get $\dot{A}_t = \dot{V}_t + \dot{M}_t^d/P_t - (\dot{P}_t/P_t)M_t^d/P_t$. Since $\dot{V}_t + \dot{M}_t^d/P_t$ must be exactly the same as the non-consumed part of direct income, i.e., the same as $r_t V_t + w_t L_t + X_t - L_t c_t = r_t(A_t - M_t^d/P_t) + w_t L_t + X_t - L_t c_t$, we then get (17.4).

Per capita accounting We convert the constraints into per capita entities in the following way. We define $a_t \equiv A_t/L_t$, implying

$$\dot{a}_t = \frac{L_t\dot{A}_t - A_t\dot{L}_t}{L_t^2} = \frac{\dot{A}_t}{L_t} - a_t n.$$

Inserting (17.5) and the definition $x_t \equiv X_t/L_t$, we get

$$\dot{a}_t = (r_t - n)a_t - (r_t + \pi_t)m_t + w_t + x_t - c_t, \qquad a_0 \text{ given.} \tag{17.6}$$

Here, the term $(r_t + \pi_t)m_t$ therefore represents the *opportunity cost* of placing part of the financial wealth in money rather than in interest-bearing assets. The No-Ponzi-Game condition (NPG) takes the form

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0, \tag{NPG'}$$

where L_0 has been eliminated. Notice that in both (17.6) and (NPG') the *growth-corrected* real interest rate, $r_t - n$, appears. On the one hand, deferring consumption gives a return through the real interest rate. But on the other hand, there will be more family members to share the return.

The problem of the household can now be formulated as follows: choose a plan $(c_t, m_t)_{t=\infty}^{\infty}$, where c_t, $m_t \geq 0$, to maximize U_0 subject to the constraints (17.6) and (NPG).

Solving the problem

The control variables are consumption c and money holding m, whereas financial wealth, a, is a state variable. The reason that m is a control variable and not a state variable is that m merely reflects a portfolio choice, given a. A discrete change in the portfolio composition, due to changed expectations, can immediately be executed.

The current-value Hamiltonian is

$$H(a, c, m, \lambda, t) = \frac{c^{1-\theta} - 1}{1-\theta} + \alpha\frac{m^{1-\varepsilon} - 1}{1-\varepsilon} + \lambda[(r-n)a - (r+\pi)m + w + x - c], \tag{17.7}$$

where λ_t is the adjoint variable associated with financial wealth per head. By the Maximum Principle an interior optimal solution will satisfy the following first-order conditions

$$\frac{\partial H}{\partial c} = c^{-\theta} - \lambda = 0 \qquad \Rightarrow c^{-\theta} = \lambda, \tag{17.8}$$

$$\frac{\partial H}{\partial m} = \alpha m^{-\varepsilon} - \lambda(r + \pi) = 0 \qquad \Rightarrow \alpha m^{-\varepsilon} = \lambda(r + \pi), \tag{17.9}$$

$$\frac{\partial H}{\partial a} = \lambda(r - n) = (\rho - n)\lambda - \dot{\lambda} \qquad \Rightarrow -\dot{\lambda}/\lambda = r - \rho. \tag{17.10}$$

In addition, the transversality condition,

$$\lim_{t\to\infty} a_t\lambda_t e^{-(\rho-n)t} = 0, \tag{TVC}$$

is necessary for optimality. And together with the first-order conditions, (TVC) is also sufficient for optimality, as shown in Appendix A. The adjoint variable λ_t can be interpreted as the shadow price (measured in current utility) of financial wealth per capita along the optimal path.

The marginal rate of substitution of consumption for holding money is, according to (17.8) and (17.9),

$$MRS_{c,m} \equiv -\frac{dc}{dm}|_{u=\bar{u}} = \frac{u_m\,(c,m)}{u_c\,(c,m)} = \frac{\alpha m^{-\varepsilon}}{c^{-\theta}} = r+\pi = i = \frac{iP}{P}, \tag{17.11}$$

where $\bar{u}$ is the instantaneous utility level obtained at (c, m). On the right-hand side of the last equality we have the price ratio of the two "goods". The nominal interest rate multiplied by P is the opportunity cost per time unit of holding P units of (nominal) money. Thus iP makes up the price of the liquidity service provided by holding cash in the amount P in one time unit. If, over the time interval $(t, t+\Delta t)$, we have m decreased by a small number, $|\Delta m|$, and bond holdings correspondingly increased, then moving along the "budget line" in the $(m\Delta t, c\Delta t)$ plane implies an increase in current consumption equal to $\Delta c\Delta t = -i\Delta t\Delta m = -i\Delta t\Delta M/(PL)$, where $\Delta M = PL\Delta m$. So $P\Delta c\Delta t = -i\Delta t\Delta M/L$. That is, the consumption spending per capita is increased by the obtained extra nominal interest income per capita, which is $|i\Delta t\Delta M/L|$. At the optimum, where the indifference curve is tangent to the budget line, the household would not be better off by such a small change in the portfolio composition.

From (17.11) we find the money demand function (conditional on the level of consumption, c) :

$$m_t = \alpha^{1/\varepsilon} c_t^{\theta/\varepsilon} i_t^{-1/\varepsilon} \equiv m^d(c_t, i_t). \tag{17.12}$$

Two features should be emphasized. First, the real money demand per head at time t is seen to be an *increasing function of* c_t. Indeed, c_t can be understood as an indicator of the volume of transactions; c_t enters with elasticity equal to θ/ε. Second, the money demand at time t is a *decreasing function of the nominal interest rate,* i_t, with absolute elasticity equal to $1/\varepsilon$.[5] Empirically, both elasticities are usually estimated to be below one, and the latter elasticity a great deal lower than the former. Goldfeld (1973) finds $1/\varepsilon \approx 0.1$. We may therefore assume ε greater than 1, which will be of importance in sections 17.3 and 17.4 below.

[5] Since c_t is endogenous for the household, we are not dealing with a "true" Walrasian demand function but a function which gives the demand for money *conditional* on the demand for consumption.

By differentiating w.r.t. t in (17.8) and inserting (17.10) we get

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho), \tag{17.13}$$

which is the traditional Keynes-Ramsey rule. If instead of c and m being separable in the instantaneous utility function $u(c,m)$ we had $u_{cm} \neq 0$, then we would not obtain the simple Keynes-Ramsey rule but a generalized version:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t, m_t)}[r_t - \rho + \eta(c_t, m_t)\frac{\dot{m}_t}{m_t}], \tag{17.14}$$

where $\theta(c,m) \equiv -cu_{cc}/u_c > 0$ and $\eta(c,m) \equiv mu_{cm}/u_c \gtrless 0$ for $u_{cm} \gtrless 0$, respectively.

The firms

The model abstracts from firms' need of cash to perform their transactions. Thus the description of the firms is as in the simple Ramsey model. The representative firm has a neoclassical production function, $Y_t = F(K_t^d, L_t^d)$, with constant returns to scale. Here, Y_t, K_t^d, and L_t^d are output, capital input, and labor input, respectively. For simplicity, technological progress is ignored and so are capital adjustment costs. We assume F satisfies the Inada conditions. Because of constant returns to scale we have $Y = F(K^d, L^d) = L^d F(k^d, 1) \equiv L^d f(k^d)$, where $k^d \equiv K^d/L^d$, and $f' > 0$, $f'' < 0$. Profit maximization under perfect competition implies

$$F_K(K^d, L^d) = f'(k^d) = r + \delta, \tag{17.15}$$

where $\delta > 0$ is the capital depreciation rate, and

$$F_L(K^d, L^d) = f(k^d) - k^d f'(k^d) = w. \tag{17.16}$$

Government/central bank

There is a consolidated government/central bank. It provides lump-sum income transfers to every household. The real transfer per household is X_t per time unit. Since the number of households (families) is normalized to 1, the government's total nominal expenditure on transfers is $P_t X_t$ per time unit. No taxes are collected and there is no issue of public debt. Hence there is *a budget deficit equal to the government expenditure* $P_t X_t$. This budget deficit is *entirely financed by the central bank "printing money"*, that is, by an expansion of the money supply.

We shall generally assume continuous equilibrium in the money market. Hence, we will use M_t to denote not only the households' money demand but also the money supply. The financing of the budget deficit can thus be written

$$\dot{M}_t = P_t X_t. \tag{17.17}$$

where $\dot{M}_t$ is the increase per time unit of the money supply. The latter equals the monetary base since there is no private banking sector. The reader may think of M as currency or liquid electronic deposits in the central bank held by the citizens. Then "printing money" means crediting citizens' accounts in the central bank.

The transfers, X_t, are thus fully financed by *seigniorage* $\dot{M}_t/P_t$, which is the revenue the public sector obtains by "printing money" (the production costs by doing so are small and can be ignored). So, in the Sidrauski model monetary policy is at the same time fiscal policy. Formally, the balance sheet of the central bank may show an accumulating outstanding account against the government. But this has no practical consequences. Or we could imagine that the government deficit is in the first instance financed by government debt issue vis-a-vis the private sector. In the next instant, the central bank buys the same amount of financial assets from the private sector. This is known as the central bank *monetizing the government debt.*

We assume that the government/central bank maintains a constant growth rate, μ, of the money supply, that is $\dot{M}_t/M_t = \mu$. This implies

$$M_t = M_0 e^{\mu t},$$

where M_0 is given. The transfer per person, $x_t \equiv X_t/L_t$, is thus endogenous and determined by

$$x_t = \frac{\dot{M}_t/P_t}{L_t} = \frac{\dot{M}_t}{M_t}\frac{M_t}{P_t L_t} = \mu m_t. \tag{17.18}$$

17.2 Equilibrium and evolution over time

General equilibrium

Clearing in the factor markets entails $K^d = K$ (i.e., the supply of real capital) and $L^d = L$ (supply of labor), in that we have normalized both the number of firms and the number of households to 1. Thus, at any time t, $k_t^d = k_t$. By substitution into the two profit-maximizing conditions above we find the equilibrium interest rate and real wage at time t as

$$r_t = f'(k_t) - \delta, \tag{17.19}$$
$$w_t = f(k_t) - k_t f'(k_t), \tag{17.20}$$

where k_t is predetermined.

Since there is no government debt and the economy is closed, the debts which households might have to each other in equilibrium balance out. Thus,

$a_t = k_t + m_t$. That both k and m are by definition non-negative has two noteworthy implications. First, the representative household satisfies automatically the NPG constraint. Second, (TVC) implies two separate aggregate transversality conditions,

$$\lim_{t\to\infty} k_t c_t^{-\theta} e^{-(\rho-n)t} = 0, \quad \text{and} \tag{17.21}$$

$$\lim_{t\to\infty} m_t c_t^{-\theta} e^{-(\rho-n)t} = 0. \tag{17.22}$$

The dynamic system

To characterize the evolution over time, we derive the fundamental differential equations of the model. From $m_t \equiv M_t/(P_t L_t)$ we get $\dot{m}_t/m_t = \dot{M}_t/M_t - \dot{P}_t/P_t - \dot{L}_t/L_t$, that is,

$$\dot{m}_t = (\mu - \pi_t - n)m_t. \tag{17.23}$$

From $k_t \equiv K_t/L_t$, it follows that $\dot{k}_t = (L_t\dot{K}_t - K_t\dot{L}_t)/L_t^2$. Inserting $\dot{K}_t = Y_t - c_t L_t - \delta K_t$, we obtain

$$\dot{k}_t = f(k_t) - c_t - (\delta + n)k_t. \tag{17.24}$$

Inserting (17.19) into (17.13) gives

$$\dot{c}_t = \frac{1}{\theta}[f'(k_t) - \delta - \rho]c_t. \tag{17.25}$$

According to (17.11) and (17.19), we have

$$\pi = \alpha m^{-\varepsilon} c^{\theta} - r = \alpha m^{-\varepsilon} c^{\theta} - f'(k) + \delta, \tag{17.26}$$

from (17.19). Thus, π_t is a function of m_t, c_t, and k_t. Consequently, with (17.26) substituted into (17.23), we have that (17.23), (17.24), and (17.25) make up three coupled differential equations in m, k, and c.

The evolution over time is determined as a solution $(m_t, k_t, c_t)_{t=0}^{\infty}$ of the coupled differential equations which satisfies: 1) k_0 equals some predetermined initial value given by history, and 2) the transversality conditions (17.21) and (17.22) hold. Whereas M_0 is predetermined, *real* per capita money holding, m_0, is, like c_0, in this model a jump variable. Indeed, $m_0 \equiv M_0/(P_0 L_0)$, where P_0 is fully flexible and adjusts instantaneously such that m_0 becomes equal to the initial real money demand, $m^d(c_0, i_0)$, given in (17.12).

That the price level is a jump variable is a crucial feature of classical and neoclassical models where prices are fully flexible even in the short run. Most economists consider this unrealistic when speaking of prices of industrial goods and services rather than prices of financial assets and some raw materials put into storage. Whether prices are fully flexible or sticky is of key importance for the short-run mechanisms of the economy. The Keynesian approach is based on the view that not the price level but the nominal interest rate is the equilibrating variable in the money market.

But suppose one accepts the classical conception of the equilibrating mechanism. Then there is the additional issue of how to rule out inflation or deflation bubbles generated by self-fulfilling expectations. An *inflation bubble* is present if sustained inflation arises due to self-fulfilling expectations. Similarly, a *deflation bubble* is present if sustained deflation arises due to self-fulfilling expectations. As we will see in Section 17.4, under certain conditions such bubbles *are* possible within the model, thus generating multiple equilibrium paths.

For now we set this problem aside and simply *assume* that neither inflation nor deflation bubbles occur. It can then be shown that the three-dimensional dynamic system, (17.23), (17.24), and (17.25), where m and c are jump variables, and k is a predetermined variable, is saddle-point stable. So there is a unique solution converging to a steady state for $t \to \infty$. The formal details are given in Appendix B. The saddle-point stability holds whether or not there is separability between c and m in the instantaneous utility function.

The model version presented here assumes this separability only to simplify. Indeed, the separability between c and m makes the dynamic system *decomposable* in the sense that the last two differential equations constitute an autonomous subsystem in k and c. This subsystem can be solved independently of (17.23). Moreover, this subsystem is identical to that of the standard Ramsey model without money, which we analyzed in Chapter 10. We know from that chapter that its solution (k_t, c_t) converges toward a steady state (k^*, c^*).

The resulting dynamics of the real money supply, m, are given by (17.23) after substitution of (17.26). And absent inflation and deflation bubbles, also m converges to a steady-state value, m^*. As mentioned, this holds also when c and m are *not* separable in the instantaneous utility function, so that (17.14), with $r_t = f'(k_t) - \delta$, replaces (17.25).

The steady state

In steady state we have $\dot{m} = \dot{k} = \dot{c} = 0$. By substitution into (17.23), (17.24) and (17.25) we get the steady-state values

$$\pi^* = \mu - n, \tag{17.27}$$

$$r^* = f'(k^*) - \delta = \rho, \tag{17.28}$$

$$k^* = f'^{-1}(\delta + \rho), \tag{17.29}$$

$$c^* = f(k^*) - (\delta + n)k^*. \tag{17.30}$$

These steady-state results also hold in the general case without separability in the instantaneous utility function. The long-run inflation rate is determined by the excess of the growth rate of the money supply over the growth rate, n, of the population which is also the output growth rate in steady state. This is as expected. The volume of transactions grow at the rate n, and to the extend money supply grows at a higher rate the outcome is inflation. The capital intensity and consumption per head in the long run are seen to depend on the rate of time preference, ρ, but not on the parameters θ, ε, and α. More impatience (higher ρ) naturally implies less capital accumulation and thereby a higher r^* and a lower k^* (in that $\partial k^*/\partial \rho = 1/f''(k^*) < 0$). The consequence is a lower c^* (since $\partial c^*/\partial k^* = f'(k^*) - (\delta + n) = \rho - n > 0$).

The steady-state value of m is obtained by inserting c^* and $i^* = r^* + \pi^* = \rho + \mu - n$ into (17.12),

$$m^* = \alpha^{1/\varepsilon} c^{*\theta/\varepsilon} (\rho + \mu - n)^{-1/\varepsilon}. \tag{17.31}$$

Firstly, we see that a higher ρ leads to a lower m^*, partly because the interest rate is increased and partly because c^*, as just observed, becomes lower. Secondly, we observe the important result that a higher monetary growth rate, μ, leads to a lower m^*. This is because the higher nominal interest rate (hence the higher opportunity costs of holding money) associated with higher inflation, $i^* = r^* + \pi^* = \rho + \mu - n$, leads to lower real money demand. On the supply side this is matched by inflation eroding the purchasing power of the money stock.

The parameters θ, ε, and α do not have any significance for the capital intensity and consumption in steady state but are seen to be concomitant determinants of m^*. Higher α implies, as expected, a higher real money demand. If we imagine a model with an explicit transaction technology, this could be interpreted as an indication of lower efficiency in the payments system and therefore a higher cash requirement. One cannot unambiguously determine the sign of the effect on m^* of a higher θ and ε.

If the model had included Harrod-neutral technological progress at the rate g, the long-run growth rate of output and consumption would be $n + g$, and (17.27) would be replaced by $\pi^* = \mu - n = \mu - n - \theta g/\varepsilon$ (see Exercise 17.1). Both θ and ε are generally estimated to be above 1. If they are close to each other, we have $\pi^* \approx \mu - n - g$, that is, long-run inflation approximately equals the excess of money growth over output growth.

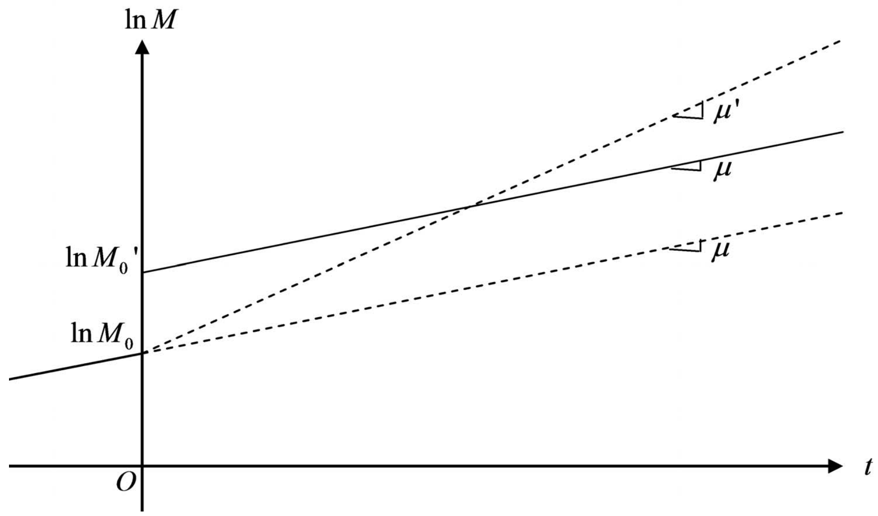

Figure 17.1: A rise in the level of the money supply versus a rise in the growth rate of the money supply.

17.3 Theoretical implications

17.3.1 Money neutrality and superneutrality

When discussing the issue of money neutrality, it is important to distinguish between two kinds of changes in monetary policy. We may think of a *shift in the level* of the money supply at a given point in time, after which the monetary growth rate remains as it was before that point in time, see Fig. 17.1. Or we may think of a *shift in the monetary growth rate* at a given point in time, after which there is a new constant growth rate, μ'.

Money is said to be *neutral* if the level and evolution of the real variables k and c are independent of the *level* of money supply. Or more precisely: let the path $(\bar{k}_t, \bar{c}_t)_{t=0}^{\infty}$ be the model's solution for the real variables k and c, given $M_0 = \bar{M}_0 > 0$. Then consider an alternative M_0, namely $M_0' = \xi \bar{M}_0$ for some arbitrary $\xi > 1$, so that $M_t = \xi \bar{M}_0 e^{\mu t}$. If the path $(\bar{k}_t, \bar{c}_t)_{t=0}^{\infty}$ is also the solution for k and c when $M_0 = M_0'$, then money is *neutral*. And this *is* evidently the case since k and c are determined solely by (17.24) and (17.25) together with the initial k and the transversality condition (17.21). Simply, a new price path $P_t' = \xi P_t$ is generated such that m_t and π_t are unaltered. Indeed, this holds not only for $\xi > 1$, but for any $\xi > 0$.

It is important that what we consider here is *not* a shift in the money supply brought about by an open-market operation where the central bank buys (or sells) interest-bearing securities from (to) the private sector. That would not only change the money supply but also the liabilities of the private sector vis-a-vis the public sector. This would imply a more complicated story.

How then can we think about the shift in the money supply? One possibility is of course to see the analysis as merely a comparison between two closed economies that are completely similar in all respects except the initial money supply. But we *can* also interpret the analysis in a more interesting way, namely as dealing with the effect of an event in historical time. If $\xi > 1$, the event is a one-off nominal *income transfer* from the government to the households at time $t = 0$, financed by a discrete amount of "money printing" and accompanied by a credible announcement that the monetary growth rate, μ, will be maintained in the future. Milton Friedman (1969) called this combined fiscal and monetary policy a "helicopter drop of money" since it is much like having a central-bank helicopter fly over the countryside spewing out money. Owing to the complete flexibility of prices in the model, the only effect is an immediate proportionate rise in the price level so as to keep the real money supply unchanged.

Now, consider an alternative kind of change in monetary policy, namely a shift in the *growth rate* of the money supply to a new constant level, cf. μ' in Fig. 17.1. Money is said to be *superneutral* if the real variables k and c in *steady state* are unaffected by such a policy shift. We see from (17.29) and (17.30) that this is in fact the case. The only steady-state effects are that the inflation rate moves one-to-one with μ and per capita real liquidity moves in the opposite direction of μ, cf. (17.31).

The mechanism behind superneutrality can be illustrated in the following way. On the one hand we have

$$\mu \uparrow \Rightarrow \pi \uparrow \Rightarrow k \uparrow \quad \text{(as a tendency)},$$

because the opportunity cost of holding money is greater when the inflation rate, and thereby the nominal interest rate, is greater. Thereby households are induced to let capital take a greater place in the portfolio. This effect is called the *Tobin effect* (after Tobin 1965).

On the other hand, we have

$$k \uparrow \Rightarrow r \downarrow \Rightarrow c \uparrow \Rightarrow k \downarrow \quad \text{(as a tendency)},$$

because greater capital intensity results in a lower marginal productivity of capital and therefore a lower real interest rate. The positive present-value effect of this (a wealth effect) stimulates consumption, while saving, and therefore capital formation, is reduced. In this way the tendency for rising capital intensity cancels itself. The result is due to the representative agent description of the household sector. This implies that the Keynes-Ramsey rule holds not only for the individual households but also at the aggregate level. This restricts the real interest rate in the long run to equal the rate of impatience, ρ, and this pins down the long-run capital intensity.

Money is said to be *super-superneutral,* if the solution path for (k_t, c_t) also *outside the steady state* is independent of the money growth rate μ. This is satisfied in *this* version of the Sidrauski model where consumption and money holdings are separable in the instantaneous utility function. Indeed, as noticed above, the differential equations (17.24) and (17.25) make up an independent system identical to the Ramsey model without money. But while the properties of neutrality and superneutrality of money hold whether $u_{cm} = 0$ (as here) or $u_{cm} \neq 0$ (as in (17.14)),[6] super-superneutrality holds only when $u_{cm} = 0$. Hence, the conclusion is that outside the steady state, the growth rate of the money supply will, in general, have an impact on capital accumulation. Simulation of the model with "realistic" parameter values indicates, however, that these effects are modest (cf. Blanchard and Fischer, 1989, p. 193).

Apart from the super-superneutrality, the neutrality results hinge neither on the imposed CRRA utility functions nor on the assumed additive separability of consumption and money holding. The superneutrality hinges on the representative agent approach and need not go through in overlapping generations models, an issue to which we return in the next chapter.

A more fundamental non-robustness problem is associated with the first mentioned money neutrality result according to which resource allocation is unaffected by a shift in the *level* of the money supply. Let us consider the supposed underlying adjustment mechanism.

Questioning the short-run adjustment mechanism

Equilibrium in financial markets requires that money demand equals money supply:

$$\begin{aligned} (\frac{M}{P})^d &= \frac{M}{P}, \qquad \text{that is,} \\ m^d(c, r + \pi^e)L &= \frac{M}{P}, \end{aligned} \tag{17.32}$$

where we have inserted the general form of the per-capita money-demand function from (17.12), but with i written as $r + \pi^e$ in order to underline the role of expectations (in spite of the assumed ex post equality of π^e and π). The model treats the current interest rate, r, as given from the real side of the economy

[6] Neutrality still holds because even in the latter case, P itself does not enter the three-dimensional dynamic system. And superneutrality still holds because (17.27) - (17.30) still hold.

(the current marginal productivity of capital at full employment) and π^e as determined by m, k, and c, as indicated by (17.26). In the long run, π^e reflects μ one to one, as indicated by (17.27).

We consider a one-off level shift in the money supply brought about by a "helicopter drop of money" followed by unchanged monetary growth at rate μ. In accordance with classical economic thinking, the model assumes that such a shift leaves inflation expectations, π^e, and real variables, like Y, c, k, and r, unchanged. The higher money supply only drives up the price level, P, and does so quickly. We may imagine that the increase in the money supply in the *very short* run results in excess supply of money. In (17.32), "=" is thereby replaced by "<". Now people respond to their excess liquidity position and their increased financial wealth by spending more on both consumption and investment. In the aggregate, however, this attempt is frustrated since aggregate production is at capacity level already. The excess liquidity is hereby transformed into excess demand for goods, triggering a rise in the general price level. As a result, the excess supply of money is eliminated by a fall in the real value per unit of money, $1/P$.

This is a manifestation of the classical perception that the general price level is perfectly flexible like a share price or an exchange rate. Empirical macroeconomics does not support this idea. It is the nominal interest rate, i, which promptly responds to a sudden change in the money supply.[7] This accords with Keynes' theory where the short-run outcome is a fall in both the nominal and the real interest rate, $r \equiv i - \pi^e$. In this line of thought less than full capacity utilization is the normal state of affairs, and the real interest rate is not tied to the marginal productivity of capital in the short run.

17.3.2 Milton Friedman's zero interest rate rule

Returning to the Sidrauski model, what can the monetary authority (government or central bank) do to make the steady state of the economy (approximately) the best possible as seen from the point of view of the representative household?

Within the frames of the model, the monetary authority controls the growth rate, μ, of the money supply. A change in μ will, as mentioned, not have an impact on k^* and c^*, but will change real money holdings, m^*, in the opposite direction, as (17.31) indicates. Thus, lower (possibly negative) growth in the *nominal* money supply leads in equilibrium to a greater *real value* of the money supply. Other things equal, this greater liquidity increases the representative household's utility. We obtain a steady state with approximately maximal welfare by setting μ close to the negative of the "effective" rate of time preference $-(\rho - n)$ (yet slightly larger

[7] For a discussion, see for example the symposium on "The Monetary Transmission Mechanism" in *Journal of Economic Perspectives*, vol. 9, no. 4, 1995.

because otherwise a steady state with $P > 0$ cannot exist according to (17.31)). With this policy the opportunity cost of holding money, the *nominal* interest rate, comes down close to nil.[8] So does the marginal utility of m, $\alpha m^*(\mu)^{-\varepsilon}$, since $m^*(\mu) \equiv \alpha^{1/\varepsilon} c^{*\theta/\varepsilon}(\mu + \rho - n)^{-1/\varepsilon}$ (cf. (17.31)) becomes extremely large, cf. Fig. 17.2.

Although the logic goes through for any positive elasticity of marginal utility of money, ε, the intuition goes through clearest in the empirically realistic case of $\varepsilon > 1$ (where the interest elasticity of money demand is $1/\varepsilon < 1$). In this case, shown in Fig. 17.2, the utility-of-money has a least upper bound, $-\alpha/(1-\varepsilon)$. By choosing μ close to $-(\rho - n)$, the monetary authority can get arbitrarily close to this upper bound. Thereby the *marginal* utility of m will be practically zero so that the gain by moving μ even closer to $-(\rho - n)$ is negligible. *Approximate* satiation with money has been obtained.[9]

We may draw a parallel to the theory of "shoe-leather costs" of inflation. High inflation encourages people to hold little cash because its value deteriorates quickly along with the rising prices in the economy. Hence, before the modern times of net banking and electronic payment, high inflation induced people to make "frequent trips to the bank" to withdraw cash. The frequent walking to the bank resulted in "shoe-leather wear and tear". To minimize these costs, which of course subsume all kinds of time and inconvenience costs associated with having to hold small amounts of cash, a low-inflation policy is recommendable.

Similarly, since the nominal interest rate is the pecuniary opportunity cost of holding money, a high nominal interest rate induces people to limit their average money holding in favor of holding interest-bearing assets. So "more frequent trips to the bank" are needed and the "shoe-leather wear and tear" becomes higher. But a zero nominal interest rate policy is capable of eliminating these "shoe-leather costs".

This is Milton Friedman's suggested long-run policy rule (from his legendary article "The Optimum Quantity of Money", 1969). The population should be satiated with liquidity – that is, *real* money. Since $\rho - n > 0$, the needed μ is negative, implying $\dot{M} < 0$. So the *nominal* money supply should gradually *decline*. This requires *negative* transfers, i.e., $X_t < 0$, which amounts to (lump-sum) *taxes* being levied so as to gradually reduce the money supply.

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[8] Indeed, if $\mu \to -(\rho - n)$, then i^* $(= r^* + \pi^* = \rho + (\mu - n)) \to 0$.

[9] Owing to the assumed CRRA functional form of the utility of money holding there is no possibility of full satiation with money. Full satiation with money requires that for some given $\bar{c} > 0$, there is an $\bar{m} > 0$ such that $u_m(\bar{c}, m) = 0$ for $m > \bar{m}$.

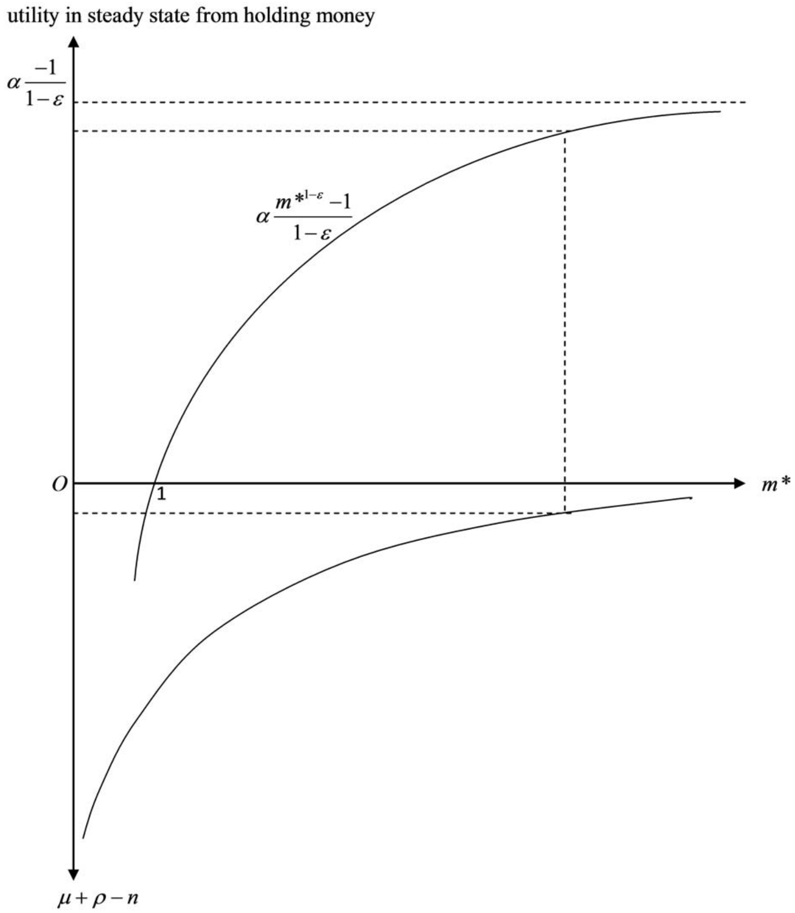

Figure 17.2: The lower is μ, hence $\mu + (\rho - n)$, the greater are real money holdings in steady state and so are the liquidity services of money. The case ϵ>1.

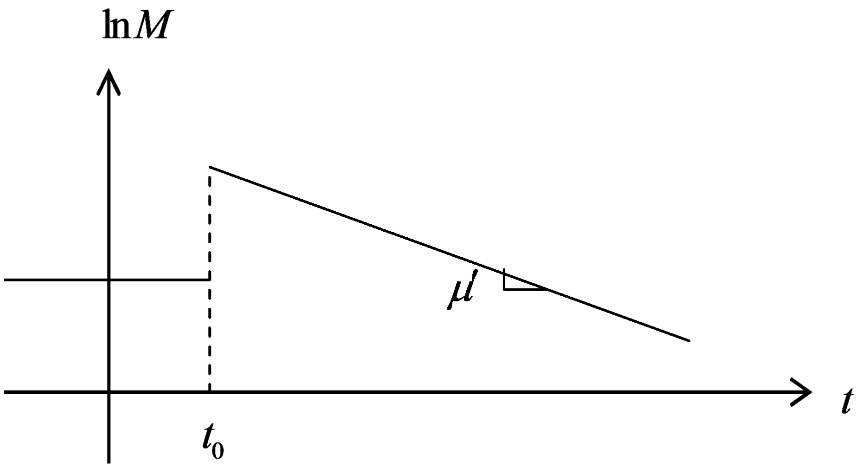

Figure 17.3: A one-off increase in the money supply as a preliminary to a shift to negative money growth.

17.3.3 Discussion

The recommendation that monetary policy should be *deflationary* is heavily disputed. And apparently no central bank has ever tried to implement the idea in practice (and in the mentioned article Friedman himself in fact expressed several reservations regarding its implementation in the short and medium run). There are several reasons for this.

First, as a preliminary to the shift to a negative growth rate of money supply, such a policy would require a foregoing *credible announcement* combined with a *one-off increase* in the money supply, as illustrated in Fig. 17.3. This is to avoid that the private sector is hit by an unforeseen drop in the price level when the deflationary policy is implemented (which could have devastating consequences for firms and households with debt contracted in nominal terms). Let us explain in more detail.

Consider an economy with $n = 0$. Suppose that before time t_0 the economy was in steady state with, say, zero inflation, $\pi^* = \mu = 0$. Then, at time t_0, μ is decreased to a level $\mu' < 0$ and credibly announced to stay there forever. According to the model, expected (and actual) inflation π^e immediately falls to $\pi^{*\prime} = \mu'$, cf. Section 17.4. So the opportunity cost of holding money is now smaller than before t_0. This results in a rise in real money demand. If no one-off increase in the money supply has taken place at time t_0, then there will be excess demand for money. The real value per unit of money, $1/P$, would then immediately go up through a drop in the price level, P. This would have unwelcome consequences for households and firms with nominal debt. To avoid such a drop in the price level, the central bank has to initiate its $\dot{M}/M < 0$ policy with a sufficient *rise* in M_{t_0}.[10] This zigzag movement may generate a credibility problem, however.

The second problem with a zero interest rate rule is that in practice it could easily lead to recession. Nominal price rigidities are crucial for the short-run working of the economy, but the Sidrauski model ignores them. In real world situations the described policy may bring about a *rise* in the nominal and real interest rate resulting in deficient aggregate demand and involuntary unemployment. Many macroeconomists believe, in contrast to Milton Friedman and his followers, that the aim of stability and flexibility calls for: 1) maintaining a *positive* though moderate rate of inflation; 2) allowing changes in the money supply to *respond* to the general state of the economy and the inflation rate. If we think in broader terms and include private banks' money creation, we are here at the key dividing line between Keynesianism and *monetarism* (the economic doctrines of Friedman). The latter has always recommended what is known as the *k-percent money growth rule* (where money is "broad money", M_2, say), irrespective of the state of the business cycle. In Friedman's earlier papers, k was a positive number in the neighborhood of the long-run output growth rate so as to maintain a low, preferably zero, rate of inflation. But in the mentioned 1969-paper the proposed policy involves, at least in the long run, a *negative* k.

17.4 Are inflation and deflation bubbles possible?

Returning to the Sidrauski model in its own right, it remains to consider whether expectations-driven hyperinflation or hyperdeflation can occur within the model. So far we have simply assumed such bubbles away.

According to the model, the regularity governing long-run inflation in the absence of inflation bubbles is that the long-run inflation rate, π^*, equals $\mu - n$. This means that high inflation arises only if money growth is high. And in fact the known historical cases of hyperinflation are all associated with extremely high money growth.[11]

If inflation bubbles are possible, however, hyperinflation can theoretically arise without high money growth merely because hyperinflation is expected; that is,

[10] Some historical cases of ending hyperinflations do in fact have similarity with this pattern (see Sargent 1982).

[11] It is not obvious where to draw the line between "high inflation" and "hyperinflation". A popular definition is that hyperinflation is present, when inflation is running at more than 50 % per month. Dornbusch et al. (2001, p. 426) suggest a lower threshold, namely 1000 % per year or 20-25 % per month.

as a *self-fulfilling expectation*. Surprisingly, even under the assumption of rational expectations (here perfect foresight) expectations-driven hyperinflation cannot generally be ruled out.[12] Moreover, *deflation* bubbles are possible as well (expectations-driven sustained deflation). On the other hand, whether and when these theoretical possibilities might become of practical importance is not well understood.

To see the possibility of these bubbles, together with the concomitant non-uniqueness of equilibrium paths, consider the dynamics of the real money supply per capita, m, as given by (17.23) after substitution of (17.26). As the economic evolution occurs in three dimensions, it is not easy to illustrate m's adjustment process graphically. We may simplify, however, by imagining that the economy's capital level and consumption level already *have* reached their steady-state values, k^* and c^*. Then by (17.23), (17.26), and (17.28) follows the one-dimensional differential equation

$$\dot{m}_t = (\mu - \alpha m_t^{-\varepsilon} c^{*\theta} + \rho - n)m_t = (\rho + \mu - n)m_t - \alpha m_t^{1-\varepsilon} c^{*\theta}. \tag{17.33}$$

We see that

$$\dot{m}_t \lesseqqgtr 0 \quad \text{for} \quad m_t \lesseqqgtr m^* = \alpha^{1/\varepsilon} c^{*\theta/\varepsilon}(\rho + \mu - n)^{-1/\varepsilon}, \tag{17.34}$$

respectively, where we have used (17.31) and that $\varepsilon > 0$.

For a given expected and actual initial inflation rate, π_0, classical equilibrium in the money market entails that the initial price level P_0 immediately adjusts such that

$$m_0 \equiv \frac{M_0^d}{P_0 L_0} = m^d(c^*, \rho + \pi_0) \lesseqqgtr m^d(c^*, \rho + \mu - n) \ = m^* \text{ for } \pi_0 \gtreqqless \pi^* = \mu - n, \tag{17.35}$$

respectively. Combining this with (17.34) entails

$$\dot{m}_0 \equiv \lim_{\Delta t \to 0^+} \frac{m(\Delta t) - m(0)}{\Delta t} \lesseqqgtr 0 \quad \text{for} \quad \pi_0 \gtreqqless \pi^* = \mu - n,$$

respectively. From this we see, first, that the inflation rate $\pi^* = \mu - n$ can be rationally expected to reign for all $t \geq 0$; if this rate is expected, m_t remains constant at the level m^* for all $t \geq 0$. Indeed, this is the case entailed in the analysis of Section 17.3 above.

The question is whether the initially expected inflation rate, π_0, *must* equal the steady-state inflation rate, π^*, to be rational. Or, what amounts to the same: is the only possible value for m_0 the steady-state value, m^*? The full answer to this depends on the parameter ε, which is inversely related to the (absolute) interest elasticity of money demand, $1/\varepsilon$.

[12] This also entails that, in the non-separable case, $u_{cm} \neq 0$, absence of superneutrality can theoretically arise from this source alone.

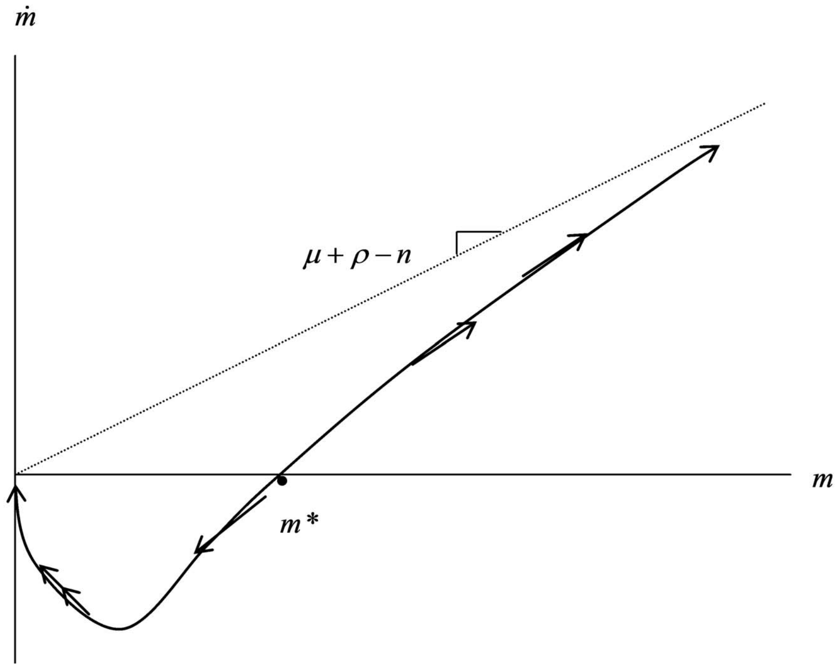

Figure 17.4: Dynamics of m when $k = k^*$ and $c = c^*$. The case $\epsilon < 1$.

The case $\varepsilon < 1$

Fig.17.4 illustrates the dynamics when the interest elasticity of money demand is above one, i.e., when $\varepsilon < 1$. The solid curve with arrows graphs the function (17.33).

Suppose $0 < m_0 < m^*$. Then, by (17.34), $\dot{m}_t < 0$ for all $t \geq 0$. We get $m_t \to 0$, reflecting that the inflation rate grows beyond all bounds, cf. (17.26) with $c = c^*$ and $k = k^*$ where $f'(k^*) + \delta = \rho$. In the process, the inflation rate is not so explosive as to go to infinity in *finite* time, however. And so the real money supply does not reach zero in *finite* time. Hence this inflationary process *can* be expected rationally; *if* it is generally expected, this expectation will be self-fulfilling. The higher the inflation rate expected, the more eager the attempt to "flee" away from money and into real assets and goods. But within the model this is impossible in the aggregate. Supplies are given and so the price level keeps rising and thereby confirms the inflationary expectations.

Note that the process is driven purely by expectations and not by M rising faster and faster; indeed, $\dot{M}/M$ is and remains equal to the constant μ. In analogy with speculative bubbles in the stock market, we are here dealing with bubbles in the real value of money or, for short, *inflation bubbles*.[13] So, in the case

[13] The background for naming these bubbles inflation bubbles is that they reflect an excess

$\varepsilon < 1$ such bubbles can be rationally expected; and as m_0 was chosen arbitrarily within the interval $(0, m^*)$, there are thereby infinitely many equilibrium paths, all divergent, that can be generated by self-fulfilling expectations. It is only when we ignore the theoretical possibility of these bubbles that the Sidrauski model's final solution is unique and necessarily converges toward the steady-state point (k^*, c^*, m^*).

Now consider the opposite situation, $m_0 > m^*$. Then, by (17.34), $\dot{m}_t > 0$ for all $t \geq 0$, and so $m_t \to \infty$. Given $\mu \geq 0$, (17.33) shows that for $t \to \infty$, not only does m rise without bound but

$$\frac{\dot{m}_t}{m_t} \to \mu + \rho - n \geq \rho - n. \tag{17.36}$$

The process considered is one of decreasing inflation which turns into *deflation* if $\rho > 0$, since the inflation rate converges toward $-\rho$, cf. (17.26) with $c = c^*$ and $k = k^*$ where $f'(k^*) + \delta = \rho$. By (17.36), $\lim_{t\to\infty} m_t c^{*-\theta} e^{-(\rho-n)t} > 0$, so that the economy-wide transversality condition (17.22) is violated. Consequently, this scenario can not be an equilibrium. Indeed, confronted with the implied rising real wealth the household would deviate in an attempt to increase its consumption (in order not to save for the sake of saving). But this would entail excess demand in the output market and push up the price level. The assumed falling inflation or even deflation is thus contradicted and cannot be expected rationally.

This argument only holds, however, as long as $\mu \geq 0$. If $\mu < 0$, real money holding will still grow, but at a rate less than $\rho - n$. And as long as $-(\rho - n) < \mu < 0$, the $m_t \to \infty$ path will not violate the transversality condition since the growth rate of m_t will be smaller than the discount rate, $\rho - n$. Deflation bubbles are thus theoretically possible in this case. This reflects that money is a fairly attractive asset when there is deflation.

The case $\varepsilon \geq 1$

Fig.17.5 illustrates the dynamics when the interest elasticity of money demand is below one, i.e., when $\varepsilon \geq 1$. The solid curve with arrows depicts the function (17.33).

Suppose, $0 < m_0 < m^*$. Then, by (17.34), $\dot{m}_t < 0$ for all $t \geq 0$. We get $m_t \to 0$, again reflecting that the inflation rate grows without bound. This time, however, m_t reaches the value 0 in *finite* time and is *still* decreasing (since, according to (17.33), $\dot{m} \to -\infty$, when $m \to 0$). This would reflect an inflationary process which "explodes" so dramatically that the real money supply passes across zero

of inflation over the "natural" or "fundamental" rate of inflation, here $\mu - n$. Another name in the literature for these hypothetical bubbles is "bubbles on money", that is, bubbles in the real value of the asset money (cf. Blanchard and Fischer 1989, Ch. 5).

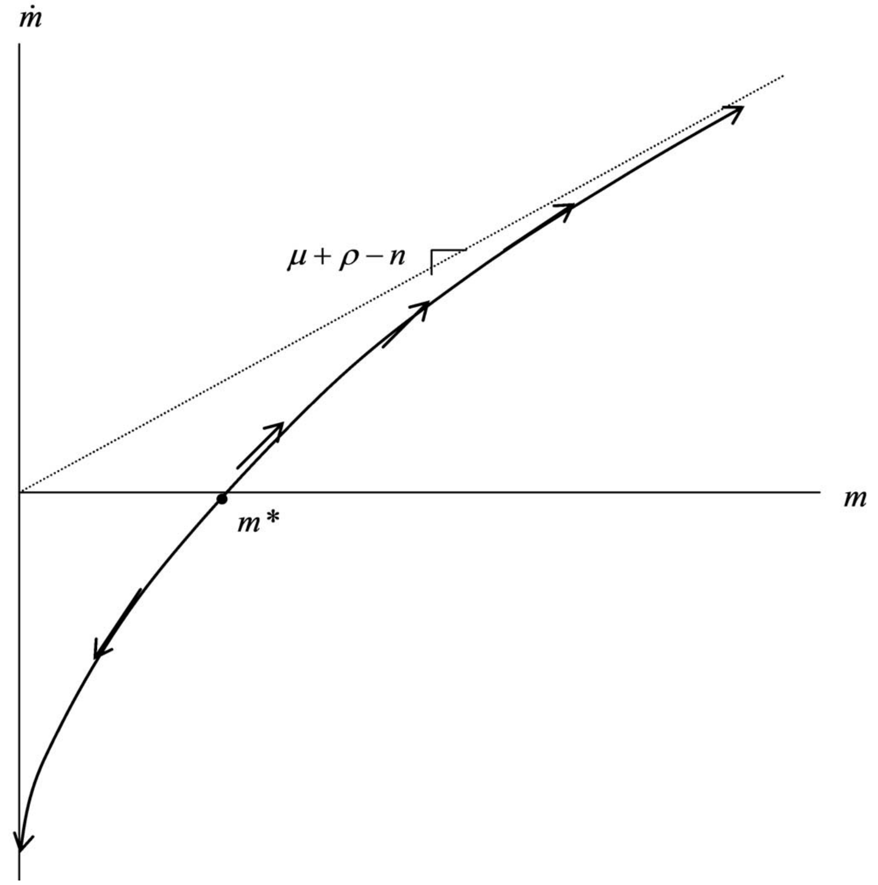

Figure 17.5: Dynamics of m when $k = k^*$ and $c = c^*$. The case $\epsilon > 1$.

in finite time. But a negative money supply is not possible, so this result cannot be rationally expected.

Conversely, assume that $m_0 > m^*$. By (17.34), $\dot{m}_t > 0$ for all $t \geq 0$ so that $m_t \to \infty$. Again, the process is one of decreasing inflation which turns into *deflation* if $\rho > 0$. Given $\mu \geq 0$, we have again not only that m_t grows without bound but also that (17.36) holds. So in this case the $m_t \to \infty$ path will violate the economy-wide transversality condition and the process can not be rationally expected.

If $\mu < 0$, however, real money holding will still grow, but at a rate less than $\rho - n$. So, as long as $-(\rho - n) < \mu < 0$, the $m_t \to \infty$ path will not violate the transversality condition. Hence, deflation bubbles are theoretically possible in this case.

The same reasoning may be applied when $\varepsilon = 1$ (logarithmic utility from money).

Is money "essential" or "inessential"?

More generally, it can be shown that expectations-driven hyperinflation can occur if and only if money is inessential. Money is defined to be *inessential* if $\lim_{m \to 0} u'_m(c,\, m)m = 0$, that is, if the marginal utility of money increases at a rate lower than the rate at which the real value of money goes to zero.[14] And money is defined as *essential,* if $\lim_{m \to 0} u'_m(c,\, m)m > 0$. In our Sidrauski model, if the (absolute) interest elasticity of money demand, $1/\varepsilon$, is at most one, then money is essential, in that we get $u'_m(c,\, m)m = \alpha m^{-\varepsilon} m = \alpha m^{1-\varepsilon}$ which, for $m \to 0$, approaches ∞, if $\varepsilon > 1$, and equals $\alpha > 0$, if $\varepsilon = 1$. However, if $0 < \varepsilon < 1$, then $\alpha m^{1-\varepsilon} \to 0$ for $m \to 0$, and money is inessential; hence, expectations-driven hyperinflation can arise in this case.

Which assumption is more reasonable, that money is essential or that it is not? Empirically, the (absolute) interest elasticity of money demand is found to be, under "normal circumstances", in a range below one. That should speak for money being essential and thus expectations-driven hyperinflation being impossible. Yet these empirical estimates need not be correct outside "normal circumstances", that is, when the real money supply becomes very small (as it does under hyperinflation).[15] From a *theoretical* point of view we should in fact *not* expect money to be essential. To show this, consider the simple case of additive utility:

$$u(c, m) = \tilde{u}(c) + \tilde{v}(m), \qquad \tilde{u}' > 0, \tilde{v}' > 0, \tilde{u}'' < 0, \tilde{v}'' < 0.$$

Money being essential implies

$$\lim_{m \to 0} \tilde{v}'(m)m > 0.$$

As Lemma 1 in Appendix C shows, this implies

$$\lim_{m \to 0} \tilde{v}(m) = -\infty.$$

This behavior of the utility function $\tilde{v}$ seems implausible, as it implies that no finite quantity of the consumption good can compensate an individual for not not holding at least *some* money, as little as it may be. This is a kind of unconditional necessity of money holding which seems inconsistent with money's role as merely reducing transaction costs (frictions) in the economy.

Hence, theoretically it seems we cannot rule out the possibility of hyperinflation driven by expectations. On the other hand, this bubble theory has at least two shortcomings. One was emphasized already in Section 17.3, namely that the short-run adjustment mechanism postulated by the Sidrauski model can hardly

[14] It is enough that this holds only for c in the relevant range.

[15] See next chapter.

be considered a good approximation to reality. (A counter argument to this, however, is that it may be a better approximation in situations where inflation is high already from the beginning.) Another shortcoming is that it is hard to tell how people's expectations in a potential bubble situation should be coordinated. Considering the law of movement for real money balances, (17.33), one should remember that the initial value, m_0, is not predetermined, but determined by the initial expected inflation rate, π_0^e. Indeed, from (17.32) and (17.12) we have

$$m_0 = \frac{M_0^d}{P_0 L_0} = m^d(c^*, r^* + \pi_0^e) = \alpha^{1/\varepsilon} c^{*\theta/\varepsilon} (r^* + \pi_0^e)^{-1/\varepsilon}.$$

Outside the "fundamental" steady state (where π^e is linked to money growth through $\pi = \mu - n$), it is hard to see how agents should agree on any particular π_0^e. In fact, there are infinitely many values of π_0^e with the property that if everybody has this expectation, then it will be self-fulfilling according to the model.

Whatever the effects of possible self-fulfilling expectations, empirical hyperinflations have been associated with immense growth in the money supply. In the next chapter the relation of this fact to large persistent government budget deficits is addressed.

17.5 Concluding remarks

(incomplete)

After the global financial crisis and the Great Recession, the topic of "helicopter money" has had a renaissance. The economic circumstances are quite distinct from the full capacity utilization assumed by the Sidrauski model. Because of the long period of lack of recovery in Europe, especially in the Eurozone, and lack of punch of conventional monetary policy because of the zero-lower-bound on the nominal interest rate, discussions about a need for money-financed expansionary fiscal policy, i.e., "helicopter money", have thrived. This theme is taken up in Chapter ...

To be thought over:

Should the exposition above be extended with government bonds and changes in the money supply through open-market operations? Cf. Buiter (2014).

The additive separability seems unrealistic and $u_{cm} > 0$ likely. Check in Walsh and Woodford 2003.

17.6 Literature notes

Comment on Sidrauski (1967b) and ...

The k-percent rule, see (?) Friedman, M., 1960, A program for monetary stability.

As to footnote 4, see Buiter and Siebert, p. 443-444.

MIU in Blanchard's OLG model with nominal rigidities, see Buiter (2014). Budget deficits financed partly by bond issue, partly by money issue. Quantitative Easing (QE) and "helicopter money".

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17.7 Appendix

A. Necessity and sufficiency of the transversality condition (TVC) together with the first-order conditions

Before considering the transversality condition, we reduce the formats of both the household's dynamic budget constraint and its intertemporal budget constraint to forms similar to those considered in Section 9.4. This will allow us to apply the general theory of that chapter.

In view of (17.6) and the identity $i_t = r_t + \pi_t$, we have

$$\dot{a}_t = (r_t - n)a_t + w_t + x_t - (c_t + i_t m_t) \equiv (r_t - n)a_t + w_t' - c_t', \qquad a_0 \text{ given}, \quad (17.37)$$

where $a_t \equiv A_t/L_t$ and we have introduced the definitions $c_t' \equiv c_t + i_t m_t$ ("broad consumption") and $w_t' \equiv w_t + x_t$. Per capita human wealth is then given as

$$h_0 = \int_0^\infty w_t' e^{-\int_0^t (r_s - n)ds} dt,$$

and the intertemporal budget constraint is

$$\int_0^\infty c_t' e^{-\int_0^t (r_s - n)ds} dt \leq a_0 + h_0. \quad \text{(IBC)}$$

The relevant solvency condition We claimed in Section 17.1 that the relevant solvency requirement in per capita terms is the condition

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0. \quad \text{(NPG')}$$

Since $a_t \equiv A_t/L_t$, that condition is equivalent to

$$\lim_{t\to\infty} A_t e^{-\int_0^t r_s ds} \geq 0, \quad \text{(NPG)}$$

which is more convenient for our present purpose. This condition allows that in the long run, the household (or "family") can still have a negative net financial position ($A < 0$), the absolute value of which is even growing over time although, at most, at a rate *less* than the interest rate.

But why is it that (NPG) is the relevant solvency requirement in the present model where we have *two* assets, money and a non-monetary asset, so that $A_t \equiv M_t/P_t + V_t$? To answer this, consider first a *finite* time horizon, T. Then, the solvency restriction is plainly that all previously accumulated debt can be repaid at the terminal date. To ensure this, the condition

$$\lim_{t\to T, t<T} A_t \equiv \lim_{t\to T, t<T} \frac{M_t}{P_t} + \lim_{t\to T, t<T} V_t \geq 0 \tag{*}$$

is sufficient. This condition allows $\lim_{t\to T, t<T} V_t < 0$, i.e., right up to time T there is a positive debt. Indeed, the debt can be cleared at the terminal date T by transferring the needed amount of cash. Indeed, ruling out discontinuity in P at time T, the needed amount of cash is available since (*) ensures that $\lim_{t\to T, t<T} M_t/P_t \geq -\lim_{t\to T, t<T} V_t > 0$.

For an *infinite* horizon, the analog to (*) is the constraint (NPG). Suppose $\lim_{t\to\infty} V_t \exp(-\int_0^t r_s ds) < 0$. Then, in view of $A_t \equiv M_t/P_t + V_t$, (NPG) ensures that $\lim_{t\to\infty}(M_t/P_t)\exp(-\int_0^t r_s ds) \geq -\lim_{t\to\infty} V_t \exp(-\int_0^t r_s ds) > 0$.

The transversality condition Above we have reduced the formats of both the household's dynamic budget constraint and its intertemporal budget constraint to forms similar to those considered in Section 9.4. This allows us to apply the general theory of that chapter.

We first prove that an admissible interior path, $(a_t, c_t, m_t)_{t=0}^{\infty}$, can only be optimal if, in addition to the first-order conditions stated in Section 17.1, it satisfies the transversality condition

$$\lim_{t\to\infty} a_t \lambda_t e^{-(\rho-n)t} = 0. \tag{TVC}$$

Even allowing for the possibility of satiation with money, the marginal utility of consumption is, by assumption, always positive. This implies that an optimal path must satisfy (IBC) with strict equality. Thereby, in view of Proposition 1 of Chapter 9, an optimal path must also satisfy (NPG) with strict equality. Consequently, by Proposition 2 of Chapter 9, (TVC) must hold as well. This proves the *necessity* of (TVC).

In Section 17.1 we also claimed that the first-order conditions, together with (TVC), are *sufficient* for an optimal solution. This claim is an immediate implication of the Mangasarian's sufficiency theorem (see Math tools) since the Hamiltonian, given in (17.7), is jointly concave in (a, c, m) for every t.

B. The transitional dynamics

In Section 17.2 we claimed that if inflation and deflation bubbles are precluded, then the Sidrauski model has a unique solution (k_t, c_t, m_t), which converges toward the steady-state point (k^*, c^*, m^*) for $t \to \infty$. We will now show this formally. The dynamic system of the model is three-dimensional:

$$\dot{k} = f(k) - c - (\delta + n)k, \tag{17.38}$$
$$\dot{c} = \frac{1}{\theta}[f'(k) - \delta - \rho]c, \tag{17.39}$$
$$\dot{m} = (\mu - \alpha m^{-\varepsilon} c^{\theta} + f'(k) - \delta - n)m, \tag{17.40}$$

where $k_0 > 0$ is given (predetermined), while c and m are jump variables. For the determination of the evolution over time we have, other than the given k_0, the requirement that the transversality conditions (17.21) and (17.22) are satisfied and that neither inflation nor deflation bubbles occur.

We compute the Jacobian matrix for the system:

$$J(k,c,m) \equiv \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial c} & \frac{\partial \dot{k}}{\partial m} \\ \frac{\partial \dot{c}}{\partial k} & \frac{\partial \dot{c}}{\partial c} & \frac{\partial \dot{c}}{\partial m} \\ \frac{\partial \dot{m}}{\partial k} & \frac{\partial \dot{m}}{\partial c} & \frac{\partial \dot{m}}{\partial m} \end{bmatrix} = \begin{bmatrix} f'(k) - \delta - n & -1 & 0 \\ \frac{1}{\theta} f''(k)c & \frac{1}{\theta}[f'(k) - \delta - \rho] & 0 \\ f''(k)m & -\alpha\theta m^{1-\varepsilon} c^{\theta-1} & j_{33} \end{bmatrix}$$

where $j_{33} = \mu - \alpha m^{-\varepsilon} c^{\theta} + f'(k) - \delta - n + \alpha\gamma m^{-\varepsilon} c^{\theta}$. Evaluated in the steady state, the Jacobian matrix becomes

$$J(k^*, c^*, m^*) = \begin{bmatrix} \rho - n & -1 & 0 \\ \frac{1}{\theta} f''(k^*)c^* & 0 & 0 \\ f''(k^*)m^* & -\alpha\theta m^{*1-\varepsilon} c^{*\theta-1} & \alpha\gamma m^{*-\varepsilon} c^{*\theta} \end{bmatrix}$$

This matrix is block-diagonal and the 2×2 sub-matrix in the upper left corner has the determinant

$$\frac{1}{\theta} f''(k^*)c^* < 0.$$

Since the determinant of a matrix is always equal to the product of the eigenvalues of the matrix, the eigenvalues of this 2×2 matrix are real and opposite in sign as J is block-diagonal. The third eigenvalue of the Jacobian matrix is simply the remaining element on the main diagonal, $\alpha\gamma m^{*-\varepsilon} c^{*\theta} > 0$.

There are thus one negative and two positive eigenvalues. Hence, the steady state of our three-dimensional dynamic system is a *saddle point*.[16] For (local)

[16] A steady-state point in $\mathbb{R}^n$ is called a *saddle point* if all eigenvalues of the Jacobian matrix of the dynamic system, evaluated in the steady state, have non-zero real parts, and at least two of the eigen values have real parts of opposite sign.

saddle-point *stability* of the steady state we further need that: (a) the number of jump variables is equal to the number of positive eigenvalues; (b) locally, the saddle path is not parallel to the linear subspace spanned by two eigenvectors corresponding to the two positive eigenvalues; (c) there is a boundary condition on the system such that the diverging paths are ruled out as solutions. The requirement (a) is satisfied since there are two jump variables, c and m. By a bit of linear algebra, the requirement (b) can be shown to hold. It would take us too far to go through the details. The requirement (c) is satisfied since, by the transversality condition of the representative household and by precluding inflation and deflation bubbles, we have ruled out that the economy can follow one of the divergent paths.

The steady state is thus (at least locally) *saddle-point stable.* In the present case this means that, for some $\beta > 0$, there is an interval $(k^* - \beta, k^* + \beta)$ around the steady-state value of k such that for any arbitrary initial value of the pre-determined variable, k_0, in this interval, there exist unique initial values of the jump variables such that the solution, $(k_t, c_t, m_t)_{t=0}^{\infty}$, of the system of differential equations converges toward the steady state for $t \to \infty$.

Saddle-point stability holds also in the general case where the instantaneous utility function is not additively separable and, thus, where the system of differential equations cannot be decomposed. See for instance Blanchard and Fischer, 1989, Chapter 4, Appendix B.

C. Lemma on essential money

As stated in Section 17.4, given the simple case of additive utility, $u(c, m) = \tilde{u}(c) + \tilde{v}(m)$, money being essential implies

$$\lim_{m \to 0^+} \tilde{v}'(m)m > 0. \tag{17.41}$$

LEMMA 1. The a utility function $\tilde{v}(m)$, with $\tilde{v}' > 0$ and $\tilde{v}'' < 0$, satisfy (17.41). Then

$$\lim_{m \to 0} \tilde{v}(m) = -\infty. \tag{17.42}$$

Proof (Obstfeld and Rogoff 1996, p. 545). Since $\tilde{v}' > 0$, $\lim_{m \to 0} \tilde{v}(m) = a_0$ exists, where possibly $a_0 = -\infty$. Now, assume (17.41) and suppose that (contrary to the assertion of the lemma) $a_0 > -\infty$. As $\lim_{m \to 0} \tilde{v}'(m)m > 0$, there exists $a_1 > 0$ and $m_0 > 0$ such that

$$\tilde{v}'(m)m > a_1 \text{ for all } m \in (0, m_0). \tag{17.43}$$

Since $\tilde{v}$ is a strictly concave function,

$$\tilde{v}(m) - a_0 > \tilde{v}'(m)m \text{ for all } m \in (0, m_0). \tag{17.44}$$

Together, (17.43) and (17.44) imply $\tilde{v}(m) - a_0 > a_1$ for all $m \in (0, m_0)$. But, since $a_1 > 0$, this contradicts that $\lim_{m \to 0} \tilde{v}(m) = a_0$. □

17.8 Exercises

17.1. Adding Harrod-neutral technical progress, cf. Section 17.2.
17.2. The case $u(c, m) = (c^{\alpha} m^{1-\alpha})^{1-\theta}/(1-\theta)$, $0 < \alpha < 1, \theta > 0$.
17.3. Adding bond-financed budget deficits, cf. Buiter 2014.