Chapter 15

Further applications of adjustment cost theory

In the previous chapter we studied how strictly convex capital installation costs affect firms' fixed capital investment and how changes in the world market interest rate affect aggregate fixed capital investment in a small open economy with perfect mobility of financial capital. In the first part of the present chapter this basic setup is extended by adding a third production factor, imported oil, and then considering the effects on the economy of an oil price shock. This includes the effects on households' aggregate consumption where the modeling of the household sector is based on the Blanchard OLG framework. The aim is not only to examine effects of an oil price shock *per se*, but also to set up a more complete accounting framework for an open economy than in earlier chapters. In the concluding remarks virtues of the OLG approach compared with the representative agent approach as modeling devices for open economies are discussed.

The strictly convex capital installation costs can be seen as an exemplification of the more general notion of strictly convex *adjustment costs*. This leads to the second part of the chapter where we apply adjustment cost theory in a dynamic analysis of the housing market from a macroeconomic point of view. The idea is that like firms' fixed capital investment, residential construction can be seen as a time-consuming activity involving strictly convex adjustment costs.

15.1 Oil price shock in a small oil-importing economy

Our focus is here on medium- and long-run effects on a small open economy (abbreviated SOE) of a supply shock in the form of a shift in the world market price of some raw material or energy, that the SOE imports. The reader may think of any imported raw material of some importance. But for concreteness we consider the imported good to be oil. This is an interesting example because of its considerable weight in many countries' imports and because of the large and sudden changes that sometimes occur in the world market price of this natural resource. In 1973-74 the real price of oil almost tripled, and in 1979-80 more than a doubling of the real price of oil took place, see Fig. 15.1.

We assume:

1. Perfect mobility of goods and financial capital across borders.
2. Domestic and foreign financial claims are perfect substitutes.
3. No mobility of labor across borders.
4. Labor supply is inelastic and constant.
5. There is no government sector and no technological progress.
6. The capital adjustment cost function $G(I, K)$ is homogeneous of degree one.
7. There is perfect competition in all markets.

Our SOE thus faces an exogenous real interest rate, r_t, given from the world financial market. For convenience, let $r_t = r$ for all $t \geq 0$, where r is a positive constant. Our analysis takes output to be supply-determined as if there is always full employment, that is, we ignore the short-term Keynesian demand effects of an oil price shock. Such effects would be due to the purchasing power of consumers being undermined by a sudden increase in the price of imported oil. We shall see that even without Keynesian effects, the overall effect of an adverse oil price shock is an economic contraction in both the short and the long run.

15.1.1 Three inputs: capital, labor, and raw material

The models in the previous chapters assumed that all output is produced in one sector using only capital and labor. We could also say that the earlier models implicitly assume that at a lower stage of production raw materials and energy are continuously produced by capital and labor, but are then immediately used up at a higher stage of production, using capital, labor, raw materials, and energy. In effect, raw materials and energy need not be treated as a separate input.

When raw materials and/or energy are imported, we have to treat them as a separate input. The technology of the representative firm in the SOE is consequently given as a three-factor production function,

$$\tilde{Y}_t = F(K_t, L_t, M_t), \quad F_i > 0, F_{ii} < 0 \quad \text{for} \quad i = K, L, M, \tag{15.1}$$

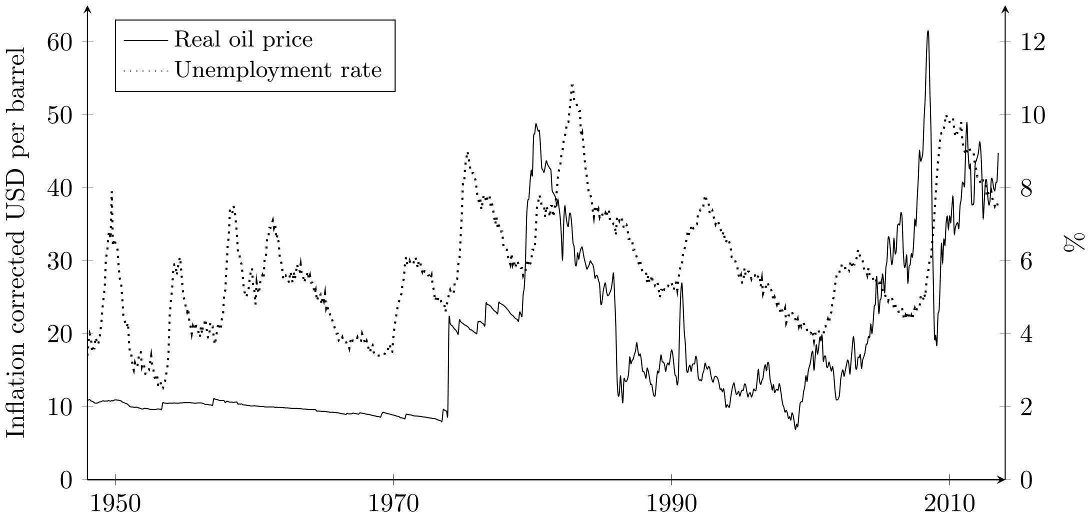

Figure 15.1: Real oil price per barrel and U.S. unemployment rate 1948-2013. Source: Bureau of Labor Statistics and Federal Reserve Bank of St. Louis.

where $\tilde{Y}$ is aggregate output gross of adjustment costs and physical capital depreciation, K is capital input, and L is labor input, whereas M is the input of the imported raw materials or energy (M for "Materials"), say oil, all measured per time unit.[1] The production function F is assumed neoclassical with CRS w.r.t. its three arguments. Thus, as usual there are positive, but diminishing marginal productivities of all three production factors. But in addition we shall need the assumption that the three inputs are *direct complements* in the sense that all the cross derivatives of F are positive:

$$F_{ij} > 0, \quad i \neq j. \tag{15.2}$$

In words: the marginal productivity of any of the production factors is greater, the more input there is of any of the other production factors.[2]

The increase per time unit in the firm's capital stock is given by

$$\dot{K}_t = I_t - \delta K_t, \qquad \delta \geq 0,$$

[1] As long as we have oil import in our mind, we should not primarily think of, for example, Denmark (even less so UK and Norway) as our case in point. Denmark has since 1996 been a net exporter of oil and natural gas. But most other European countries will fit as good examples.

[2] For a two-factor neoclassical production function with CRS we always have direct complementarity, i.e., $F_{12} > 0$. But with more than two production factors, direct complementarity for all pairs of production factors is not assured. Therefore, in general, direct complementarity is an additional assumption. However, the Cobb-Douglas function, $Y = K^{\alpha_1} L^{\alpha_2} M^{1-\alpha_1-\alpha_2}$, where $\alpha_i > 0$, $i = 1, 2$, and $\alpha_1 + \alpha_2 < 1$, automatically satisfies all the conditions in (15.2).

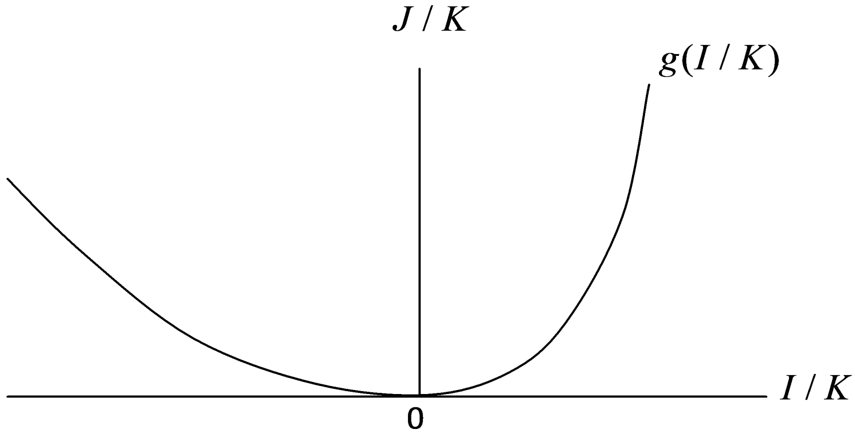

Figure 15.2: Capital adjustment cost per unit of installed capital.

where I is gross investment per time unit and δ is the rate of physical wearing-down of capital (physical depreciation in this model has to be distinguished from economic depreciation, cf. Section 15.1.3 below).

The firm faces strictly convex capital installation costs. Let these installation costs (measured in units of output) at time t be denoted J_t and assume they depend only on the level of investment and the existing capital stock; that is $J_t = G(I_t, K_t)$. The installation cost function G is assumed homogeneous of degree one so that we can write

$$J = G(I, K) = G(\frac{I}{K}, 1)K \equiv g(\frac{I}{K})K, \tag{15.3}$$

where the function g is *strictly convex* and satisfies

$$g(0) = 0, g'(0) = 0 \text{ and } g'' > 0. \tag{15.4}$$

The graph of g is shown in Fig. 15.2.

Gross domestic product (value added) at time t is

$$GDP_t \equiv \tilde{Y}_t - J_t - p_M M_t, \tag{15.5}$$

where p_M is the real price of oil, this price being exogenous to the SOE. For simplicity we assume that this price is a constant, but it may shift to another level (i.e., we use p_M as a shift parameter).

The decision problem of the firm

Let cash flow (before interest payments) at time t be denoted R_t. Then

$$R_t \equiv F(K_t, L_t, M_t) - g(\frac{I_t}{K_t})K_t - w_t L_t - p_M M_t - I_t, \tag{15.6}$$

where w_t is the real wage. The decision problem, as seen from time 0, is to choose a plan $(L_t, M_t, I_t)_{t=0}^{\infty}$ to maximize the market value of the firm,

$$V_0 = \int_0^{\infty} R_t e^{-rt} dt \quad \text{s.t. (15.6), and} \tag{15.7}$$

$$L_t \geq 0, M_t \geq 0, I_t \text{ free}, \qquad \text{(i.e., no restriction on } I_t) \tag{15.8}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 \text{ given}, \tag{15.9}$$

$$K_t \geq 0 \text{ for all } t. \tag{15.10}$$

To solve the problem, we use the Maximum Principle. The problem has three control variables, L, M, and I, and one state variable, K. We set up the current-value Hamiltonian:

$$\mathbf{H}(K, L, M, I, q, t) \equiv F(K, L, M) - g(\frac{I}{K})K - wL - p_M M - I + q(I - \delta K), \tag{15.11}$$

where q_t is the adjoint variable associated with the dynamic constraint (15.9). For each $t \geq 0$ we maximize the Hamiltonian w.r.t. the control variables: $\partial H/\partial L = F_L(K, L, M) - w = 0$, i.e.,

$$F_L(K, L, M) = w; \tag{15.12}$$

$\partial \mathbf{H}/\partial M = F_M(K, L, M) - p_M = 0$, i.e.,

$$F_M(K, L, M) = p_M; \tag{15.13}$$

and $\partial \mathbf{H}/\partial I = -g'(\frac{I}{K}) - 1 + q = 0$, i.e.,

$$1 + g'(\frac{I}{K}) = q. \tag{15.14}$$

Next, we partially differentiate w.r.t. the state variable, K, and equates this derivative to $rq_t - \dot{q}_t$, since r is the discount rate in (15.7):

The Maximum Principle now says that an interior optimal path (K_t, L_t, M_t, I_t) satisfies that there exists an adjoint variable q_t such that for all $t \geq 0$, the conditions (15.12), (15.13), (15.14), and (**??**) hold along the path, and the transversality condition,

$$\lim_{t \to \infty} K_t q_t e^{-rt} = 0, \tag{15.15}$$

is satisfied.

The only new optimality condition compared to the previous chapter is (15.13) which just says that optimality requires equalizing the marginal productivity of the imported input to its real price, p_M. By (15.14), the adjoint variable, q_t, can be interpreted as a *shadow price* (measured in current output units) of installed capital along the optimal path. That is, q_t represents the value to the firm of the marginal unit of installed capital at time t along the optimal path. The transversality condition says that the present value of the stock of installed capital "left over" at infinity must be vanishing.

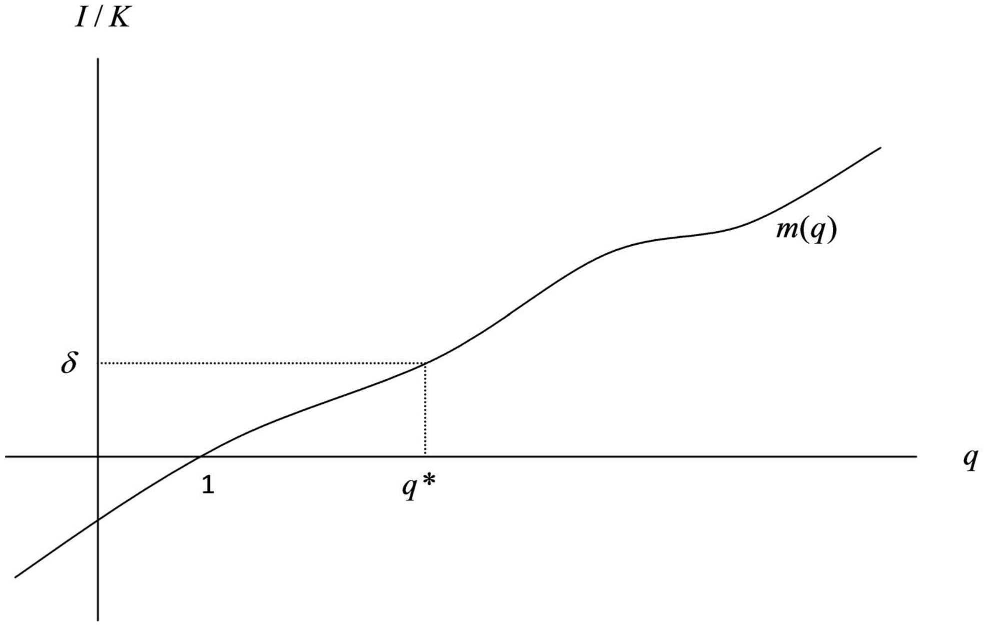

Figure 15.3

The implied investment function

Since $g'' > 0$, the optimality condition (15.14) implicitly defines the optimal investment ratio, I/K, as a function of the shadow price q,

$$\frac{I_t}{K_t} = m(q_t), \quad \text{where } m(1) = 0 \text{ and } m' = 1/g'' > 0. \tag{15.16}$$

This in the investment function of the representative firm. An example is illustrated in Fig. 15.3.

To see what the optimality condition (**??**) implies, notice that

$$\begin{aligned} \frac{\partial\left[g(\frac{I}{K})K\right]}{\partial K} &= g(\frac{I}{K}) + Kg'(\frac{I}{K})\frac{-I}{K^2} = g(\frac{I}{K}) - g'(\frac{I}{K})\frac{I}{K} \\ &= g(m(q)) - g'(m(q))m(q) = g(m(q)) - (q-1)m(q) \end{aligned}$$

from (15.14) and (15.16). Insert this into (**??**) to get

$$\dot{q}_t = (r+\delta)q_t - F_K(K_t, L_t, M_t) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{15.17}$$

By reordering, this can be written as a no-arbitrage condition,

$$\frac{F_K(K_t, L_t, M_t) - [g(m(q_t)) - (q_t-1)m(q_t)] - \delta q_t + \dot{q}_t}{q_t} = r, \tag{15.18}$$

saying that, the rate of return on the marginal unit of installed capital must equal the (real) interest rate.

To simplify the expression for the marginal productivity of capital in the differential equations (15.17), we shall now invoke some general equilibrium conditions.

15.1.2 General equilibrium and dynamics

We assume households' behavior is as described by a simple Blanchard OLG model (without retirement). Yet, in the general equilibrium of the SOE, firms' choices are independent of households' consumption/saving behavior, the analysis of which we therefore postpone to Section 15.1.4.

Clearing in the labor market implies that employment, L_t, equals the exogenous constant labor supply, $\bar{L}$, for all $t \geq 0$. In view of the convex installation costs, K_t is given in the short run and changes only gradually. We now show that the demand for oil, the market clearing wage, and the marginal productivity of capital all can be written as functions of K_t and p_M (for fixed $\bar{L}$).

First, since $F_{MM} < 0$, the firm's optimality condition (15.13) determines oil demand, M_t, as an implicit function of K_t, p_M, and $\bar{L}$:

$$M_t = M(K_t, p_M), \qquad M_K = \frac{-F_{MK}}{F_{MM}} > 0, M_{p_M} = \frac{1}{F_{MM}} < 0, \tag{15.19}$$

where the exogenous constant $\bar{L}$ has been suppressed as an argument, for simplicity. The alleged signs on the partial derivatives are implied (see Appendix A) by the standard assumption $F_{MM} < 0$ and the assumption of direct complementarity: $F_{MK} > 0$.

Second, by inserting (15.19) and $L_t = \bar{L}$ in the optimality condition (15.12), we find an expression for the real wage,

$$w_t = F_L(K_t, \bar{L}, M(K_t, p_M)) \equiv w(K_t, p_M), \qquad w_K > 0, w_{p_M} < 0. \tag{15.20}$$

The alleged signs on the partial derivatives are implied (see Appendix A) by the direct complementarity assumptions $F_{LK} > 0$ and $F_{LM} > 0$.

Third, in view of (15.19) and $L_t = \bar{L}$ we can can simplify the expression for the marginal productivity of capital:

$$F_K(K_t, \bar{L}, M(K_t, p_M)) \equiv MPK(K_t, p_M), \quad MPK_K < 0, MPK_{p_M} < 0, \tag{15.21}$$

The label MPK for this function comes from "Marginal Productivity of K". The alleged sign on the first mentioned partial derivative is implied by F being neoclassical with non-increasing returns to scale combined with the input factors being complementary (see Appendix A). That $MPK_{p_M} < 0$ follows from $F_{MM} < 0$ and $F_{KM} > 0$.

Dynamics of the capital stock

We have thus established that even when the effect of increased K on oil input is taken into account, increased K implies lower marginal productivity of capital. By implication, the analysis of the dynamics of the capital stock is completely similar to that in Chapter 14.3. Indeed, inserting (15.21) into (15.17), we get

$$\dot{q}_t = (r+\delta)q_t - MPK(K_t, p_M) + g(m(q_t)) - m(q_t)(q_t - 1), \tag{15.22}$$

where we have applied Lemma 1 of Chapter 14.1.3. Since r and p_M are exogenous, this is a differential equation with the capital stock, K, and its shadow price, q, as the only endogenous variables. Another differential equation with these two variables being endogenous can be obtained by inserting (15.16) into (15.9) to get

$$\dot{K}_t = (m(q_t) - \delta)K_t. \tag{15.23}$$

Fig. 15.4 shows the phase diagram for these two coupled differential equations. We have (suppressing, for convenience, the explicit time subscripts)

$$\dot{K} = 0 \quad \text{for} \quad m(q) = \delta, \text{ i.e., for } q = q^*,$$

where q^* is defined by the requirement $m(q^*) = \delta$. Notice, that this implies $q^* > 1$ when $\delta > 0$. We see that

$$\dot{K} \gtrless 0 \text{ for } m(q) \gtrless \delta, \text{ respectively, i.e., for } q \gtrless q^*, \text{ respectively.}$$

This is illustrated by the horizontal arrows in Fig. 15.4.

From (15.22) we have $\dot{q} = 0$ for

$$0 = (r+\delta)q - MPK(K, p_M) + g(m(q)) - m(q)(q-1). \tag{15.24}$$

If, in addition $\dot{K} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta$), this gives $0 = (r+\delta)q^* - MPK(K, p_M) + g(\delta) - \delta(q^* - 1)$ or

$$rq^* = MPK(K, p_M) - g(\delta) - \delta, \tag{15.25}$$

where the right-hand-side is decreasing in K, in view of $MPK_K < 0$ (see (15.21)). Hence, there exists at most one value of K such that the steady state condition (15.25) is satisfied.[3] This value is called K^*, corresponding to the steady state, point E, in Fig. 15.4.

As in Chapter 14.3, we end up with a phase diagram as in Fig. 15.4, where the steady state is saddle-point stable. The question now is: what is the slope of

[3] Assuming that F satisfies the Inada conditions, such a value *does* exist.

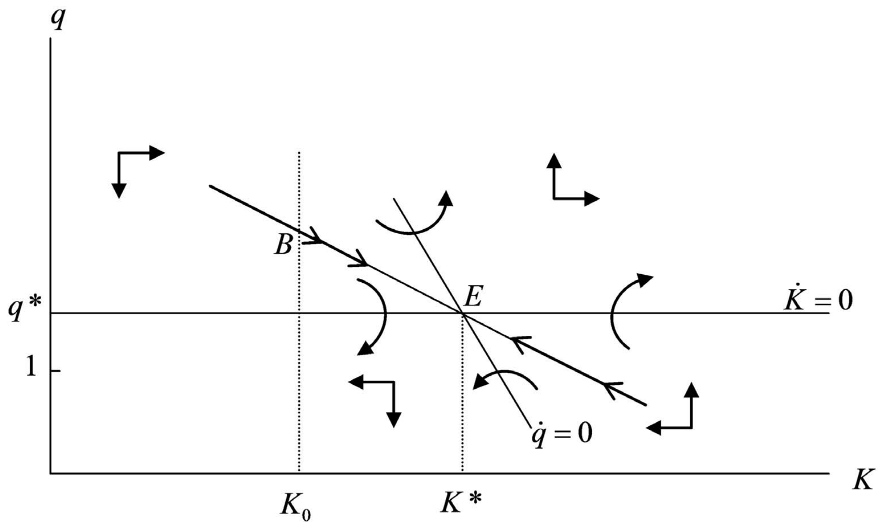

Figure 15.4

the $\dot{q} = 0$ locus? In the appendix of the previous chapter it was shown that at least in a neighborhood of the steady state point E this slope is negative, in view of $MPK_K < 0$ and the assumption $r > 0$. From (15.22) we see that

$$\dot{q} \lesseqgtr 0 \text{ for points to the left and to the right, respectively, of the } \dot{q} = 0 \text{ locus,}$$

since $MPK_K < 0$. The vertical arrows in Fig. 15.4 show these directions of movement.

Altogether the phase diagram shows that the steady state, E, is a saddle point, and since there is one predetermined variable, K, and one jump variable, q, and the saddle path is not parallel to the jump variable axis, this steady state is saddle-point stable. We can exclude the divergent paths by appealing to the representative firm's necessary transversality condition (15.15). Hence, a movement along the saddle path towards the steady state is the unique solution for the path of the capital stock and the shadow price of installed capital.

Effect of an oil price shock

Assume that until time 0, the economy has been in the steady state E in Fig. 15.4. Then, an unexpected shift in the world market price of oil occurs so that the new price is a constant $p_M' > p_M$ (and is expected to remain for ever at this level). From (15.23) we see that q^* is not affected by this shift, hence, the $\dot{K} = 0$ locus is not affected. But the $\dot{q} = 0$ locus shifts downward, in view of $MPK_{p_M} < 0$. Indeed, to offset the fall of MPK when p_M increases, a lower K is required, given q.

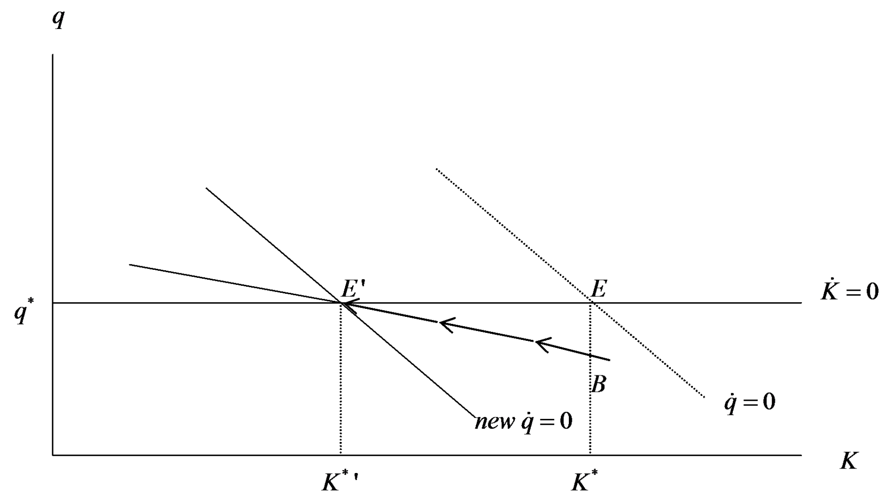

Figure 15.5

Fig. 15.5 illustrates the situation for $t \geq 0$. At time $t = 0$ the shadow price q jumps down to a level corresponding to the point B in Fig. 15.5. This is because the cost of oil is now higher, reducing current and future optimal input of oil and therefore (by complementarity) reducing also the current and future marginal productivity of capital. As a result, the value to the firm of the marginal unit of capital is immediately diminished, implying a diminished incentive to invest. Hence, gross investment jumps to a lower level not sufficient to make up for the wearing-down of capital. The capital stock decreases gradually. But this implies increasing marginal productivity of capital, hence, increasing q, and the economy moves along the new saddle path and approaches the new steady state E ' as time goes by.

This is where we see the crucial role of strictly convex capital installation costs. If these costs were not present, the model would lead to the counterfactual prediction that the new steady state would be attained instantaneously when the oil price shock occurs.

Notice, however, an important limitation of the theory. In a Keynesian short-run perspective, where firms solve a cost minimization problem for a given desired level of output (equal to the demand faced by the firms), the increase in the price of oil leads to less demand for oil, but *more* demand for capital equipment (a pure substitution effect). Hence, in the real world we may observe a fall in q^a (due to higher production costs) at the same time as investment increases, contrary to what the q-theory of investment predicts under perfect competition.[4]

[4] This is where we see the crucial role of strictly convex capital installation costs. If these costs were not present, the model would lead to the counterfactual prediction that the new

The reader should recognize that to determine the investment dynamics of the SOE we did not need to consider the households' saving decision. Indeed, one of the convenient features of the SOE model is that it can be solved *recursively:* the total system can be decomposed into an investment subsystem, describing the dynamics of physical capital, and a saving subsystem, describing the dynamics of human wealth and financial wealth of households, H and A, respectively. Although the total system has five endogenous variables, K, q, H, A, and C, the dynamics of K and q are determined by (15.23) and (15.22) independently of the other variables. Thus, (15.23) and (15.22) constitute a *self-contained subsystem of zero order.* We shall soon see that, given the solution of this subsystem of zero order, the dynamics of H are determined in a subsystem of first order in the causal ordering, and, given this determination, the dynamics of A are determined in a subsystem of second order in the causal ordering. Finally, given the determination of H and A, the path of C is determined in a subsystem of third order.

Before turning to household behavior, however, some remarks on national income accounting for this open economy with capital installation costs may be useful.

15.1.3 National income accounting for an open economy with capital installation costs

We ignore the government sector, and therefore national wealth is identical to aggregate private financial wealth, which is here, as usual, called A. We have, by definition,

$$A = V + A_f,$$

where V is the market value of firms and A_f is net foreign assets (financial claims on the rest of the world).[5] Sometimes, it is more convenient to consider net foreign debt, $NFD \equiv -A_f$, so that $A = V - NFD$. As usual, we define q_a ("average q") as the ratio of the market value of firms to the replacement cost of the capital stock,

$$q_a \equiv \frac{V_t}{K_t}.$$

steady state would be attained instantaneously when the oil price shock occurs.

Notice, however, an important limitation of the theory. In a Keynesian short-run perspective, where firms solve a cost minimization problem for a given desired level of output (equal to the demand faced by the firms), the increase in the price of oil leads to less demand for oil, but *more* demand for capital equipment (a pure substitution effect). Hence, in the real world we may observe a fall in "average q" (due to higher production costs) at the same time as investment increases, contrary to what the q-theory of investment predicts under perfect competition.

[5] Housing wealth and land are ignored.

Hence, national wealth can be written

$$A = q_a K + A_f, \tag{15.26}$$

The current account surplus is

$$\dot{A}_f = NX + rA_f, \tag{15.27}$$

where NX is net export of goods and services, also called the trade surplus. We have

$$NX \equiv GDP - C - I = \tilde{Y} - J - p_M M - C - I, \tag{15.28}$$

by (15.5).

Now, look at the matter from the income side rather than the production side. Gross national income, also called gross national product, GNP, is generally defined as the gross income of inputs owned by residents of the home country, i.e., $GNP \equiv GDP + rA_f + wL_f$. Here $rA_f + wL_f$ is total net factor income earned in other countries by residents of the home country, the first term, rA_f, being net capital income from abroad, while the second term, wL_f, represents labor income earned in other countries by residents of the home country minus labor income earned in the home country by residents in the rest of the world. Our present model ignores mobility of labor so that $wL_f = 0$. Hence,

$$GNP = GDP + rA_f. \tag{15.29}$$

At the theoretical level net national product, NNP, is defined, following Hicks (1939), as that level of consumption which would leave financial wealth, A, unchanged. We shall see that this is equivalent to defining NNP as GNP minus *economic* depreciation, D, that is,

$$NNP = GNP - D. \tag{15.30}$$

We have

$$D \equiv I - I_n, \tag{15.31}$$

where I is domestic gross investment, whereas I_n is domestic net investment in the following *value* sense:[6]

$$I_n \equiv \frac{d(q_a K)}{dt} = q_a \dot{K} + \dot{q}_a K. \tag{15.32}$$

[6] Net investment in a *physical* sense is $\dot{K} = I - \delta K$, since δ is the rate of physical wearing-down of capital.

To check whether this is consistent with the Hicksian definition of NNP, insert (15.31) and (15.32) into (15.30) to get

$$\begin{aligned} NNP &= GNP - I + \frac{d(q_a K)}{dt} = GDP + rA_f - I + \frac{d(q_a K)}{dt} \quad \text{(by (15.29))} \\ &= C + NX + rA_f + \frac{d(q_a K)}{dt} \quad \text{(by (15.28))} \\ &= C + \dot{A}_f + \frac{d(q_a K)}{dt} \quad \text{(by (15.27))} \\ &= C + \dot{A}, \end{aligned} \tag{15.33}$$

where the last equality follows from (15.26)). This *is* consistent with the theoretical definition of NNP as the level of consumption which would leave financial wealth, A, unchanged.

From (15.33) we get

$$\dot{A} = NNP - C \equiv S_n, \tag{15.34}$$

where S_n is aggregate net saving. This is consistent with the standard definition of aggregate gross saving as $S \equiv GNP - C$, since

$$\begin{aligned} S_n &\equiv NNP - C = GNP - D - C \quad \text{(by (15.30))} \\ &\equiv S - D. \end{aligned}$$

Observe also that

$$\begin{aligned} S &\equiv S_n + D = GNP - C = GDP + rA_f - C \quad \text{(by (15.29))} \\ &= rA_f + NX + I \quad \text{(by (15.28))} \\ &= \dot{A}_f + I. \quad \text{(by (15.27) and (15.31))} \end{aligned} \tag{15.35}$$

So we end up with the national accounting relationship that the current account surplus, $\dot{A}_f$, is the same as the excess of saving over domestic investment, $S - I$.

Finally, in a steady state with $\dot{A} = 0$ and $d(q_a K)/dt = 0$, (15.26) gives $\dot{A}_f = 0$. Hence, by (15.27), we have

$$NX = -rA_f \tag{15.36}$$

in the steady state, so that (in this model without economic growth) net exports exactly matches interest payments on net foreign debt, $-A_f$.

15.1.4 Household behavior and financial wealth

As already mentioned, households are described as in the simple Blanchard OLG framework with constant population, no retirement, no technical progress, and no government sector. Hence, aggregate consumption at time t is

$$C_t = (\rho + \mu)(A_t + H_t), \tag{15.37}$$

where $\rho \geq 0$ is the pure rate of time preference, and $\mu > 0$ is the mortality rate (here equal to the birth rate, since $n = 0$). Human wealth, H_t, is the present discounted value of future labor income of those people who are alive at time t, that is,

$$H_t = \int_t^\infty w_\tau \bar{L} e^{-(r+\mu)(\tau-t)} d\tau = \int_t^\infty w(K_\tau, p_M) \bar{L} e^{-(r+\mu)(\tau-t)} d\tau, \tag{15.38}$$

in view of (15.20). Inserting the solution for $(K_\tau)_{\tau=t}^\infty$, found above, (15.38) gives the solution for $(H_t)_{t=0}^\infty$. Notice, that whatever the initial value of K, we know from Section 15.1.2 above that $K_t \to K^*$ for $t \to \infty$. Applying this on (15.38) we see that, for $t \to \infty$,

$$H_t \to \int_t^\infty w(K^*, p_M) \bar{L} e^{-(r+\mu)(\tau-t)} d\tau = \frac{w(K^*, p_M)\bar{L}}{r+\mu} \equiv H^*. \tag{15.39}$$

In view of perfect competition and that the production function F and the capital installation cost function G are homogeneous of degree one, we know from Hayashi's theorem that "average q" = "marginal q", i.e., $q_a = q$ $(= \partial V^*/\partial K_t)$.[7] Therefore, by (15.26), national wealth can be written

$$A = qK + A_f. \tag{15.40}$$

Wealth and consumption dynamics

Observe that

$$\begin{aligned} GDP &= \tilde{Y} - p_M M - J = F(K, \bar{L}, M) - F_M(K, \bar{L}, M)M - J \\ &\qquad\qquad\qquad\qquad \text{(by (15.5) and (15.13))} \\ &= F_K(K, \bar{L}, M)K + F_L(K, \bar{L}, M)\bar{L} - J \\ &= F_K(K, \bar{L}, M)K + w\bar{L} - J, \qquad \text{(by (15.12))} \end{aligned} \tag{15.41}$$

where the second equality comes from Euler's Theorem applied to the CRS function $F(K, \bar{L}, M)$.

From (15.34) we have

$$\begin{aligned} \dot{A} &= S_n = NNP - C = GNP - D - C \qquad \text{(by (15.34) and (15.30))} \\ &= GDP + rA_f - D - C \qquad \text{(by (15.29))} \\ &= F_K(K, \bar{L}, M)K + w\bar{L} - J - (I - I_n) + rA_f - C \text{ (by (15.41) and (15.31))} \\ &= F_K(K, \bar{L}, M)K + w\bar{L} - J - (I - q\dot{K} - \dot{q}K) + rA_f - C \end{aligned}$$

[7] See the previous chapter. Hayashi's theorem is valid also when, as here, there are three (or more) production factors.

by (15.32) and the fact that q_a equals q. Continuing, we have

$$
\begin{aligned}
\dot{A} &= F_K(K,\bar{L},M)K + w\bar{L} - J + (q-1)I - \delta qK + \dot{q}K + rA_f - C \text{ (by (15.9))} \\
&= \left[F_K(K,\bar{L},M) - g(m(q)) + m(q)(q-1) - \delta q + \dot{q}\right] K + rA_f + w\bar{L} - C \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (15.3) and (15.16))} \\
&= rqK + rA_f + w\bar{L} - C, \qquad \text{(by the no-arbitrage condition (15.18))} \\
&= rA + w\bar{L} - C. \qquad\qquad\qquad \text{(by (15.40))}.
\end{aligned}
$$

Comparing this with (15.26) we see that in equilibrium, $NNP = r(qK+A_f)+w\bar{L}$. That is, national income is equal to the sum of income from financial wealth and income from labor, as expected. The rate of return on financial wealth is given from the world capital market, and the pay of labor is the market clearing real wage in the SOE.

Using (15.20) and (15.37), our differential equation for financial wealth can be written

$$\dot{A}_t = (r - \rho - \mu)A_t + w(K_t, p_M)\bar{L} - (\rho + \mu)H_t. \tag{15.42}$$

Since initial national wealth, A_0, is historically given, and the paths of K_t and H_t have already been determined, this differential equation determines uniquely the path of national wealth, A_t.

Suppose

$$\rho + \mu > r, \tag{15.43}$$

that is, we are *not* in the case of "very low impatience".[8] Then (15.42) implies stability of A_t so that, for $t \to \infty$,

$$A_t \to \frac{w(K^*, p_M)\bar{L} - (\rho + \mu)H^*}{\rho + \mu - r} = \frac{(r - \rho)H^*}{\rho + \mu - r} = \frac{(r - \rho)w(K^*, p_M)\bar{L}}{(\rho + \mu - r)(r + \mu)} \equiv A^*, \tag{15.44}$$

where we have used (15.39).

Finally, given the solution for H_t and A_t, (15.37) shows the solution for C_t. When the stability condition (15.43) holds, we have, for $t \to \infty$,

$$C_t \to (\rho + \mu)(A^* + H^*) = (\rho + \mu)\frac{\mu w(K^*, p_M)\bar{L}}{(\rho + \mu - r)(r + \mu)} \equiv C^*. \tag{15.45}$$

Given the stability condition (15.43), the steady-state value of national wealth in (15.44) is positive, if and only if $r - \mu < \rho < r$. This is the case of "medium

[8]Otherwise, i.e., if $\rho \leq r - p$, no steady state would exist (see (15.45) below)) and the SOE would grow *large* in the long run. Then the world market interest rate r could no longer be considered independent of what happens in this economy.

impatience" where our SOE has a degree of impatience, ρ, that is *not* vastly different from that of the "average country" in the world economy.[9]

If on the other hand our SOE is *very* impatient ($\rho > r$), then, even supposing that initial national wealth is positive, so that interest income is positive, the economy consumes more than it earns so that net saving is negative and national wealth decreases over time. Indeed, we know from the Blanchard model that the change in aggregate consumption per time unit is given by

$$\dot{C}_t = (r - \rho)C_t - \mu(\rho + \mu)A_t,$$

so that, with $\rho > r$, we get $\dot{C}_t < 0$, at least as long as $A_t \geq 0$. The economy ends up with negative national wealth in the long run, as shown by (15.44). This entails a net foreign debt over and above the market value, q^*K^*, of the firms:

$$-A_f^* = q^*K^* - A^* = q^*K^* - \frac{(r-\rho)H^*}{\rho + \mu - r} > q^*K^*. \tag{15.46}$$

This is theoretically possible in view of the fact that the economy still has its human wealth, H, as a source of income. Indeed, as long as (15.43) holds, a steady state with $A^* + H^* > 0$ exists, as indicated by (15.45).

What (15.46) shows is that a very impatient country asymptotically mortgages all of its physical capital and part of its human capital. This is a counterfactual prediction, and below we return to the question why such an outcome is not likely to occur in practice.

Intertemporal interpretation of current account movements

Finally, the level of net exports is

$$\begin{aligned} NX &= \tilde{Y} - p_M M - J - I - C & &\text{(by (15.28) and (15.5))} \\ &= F_K(K, \bar{L}, M)K + w\bar{L} - J - I - C & &\text{(by (15.41))} \\ &= \dot{A} - I_n - rA_f. & &\text{(by the third row in the derivation of } \dot{A} \text{ above)} \end{aligned}$$

In steady state, $\dot{A} = 0 = I_n$, hence,

$$\begin{aligned} NX^* &= -rA_f^* \qquad\qquad (15.47) \\ &= -r\left[q^*K^* - \frac{(r-\rho)w(K^*, p_M)\bar{L}}{(\rho+\mu-r)(r+\mu)}\right]. \qquad \text{(from (15.46))} \end{aligned}$$

[9] If all countries can be described by the simple Blanchard model, then the interest rate r in the world market is somewhat larger than the pure rate of time preference of the "average country", cf. Chapter 12.

This determines the long-run level of net exports as being equal to the interest payments on net foreign debt, $D \equiv -A_f$, so that in the steady state, the current account deficit, $rD - NX$, is zero. As expected, the economy remains solvent. In fact, the consumption function (15.37) of the Blanchard model is derived under the constraint that solvency, through a NPG condition on the long-run path of financial wealth (or debt), is satisfied.

Whatever the size relation between ρ and r, it is not necessary for equilibrium that net foreign debt is zero in the long run. Necessary in this model, which is without economic growth, is that in the long run net foreign debt is constant, i.e., the current account is ultimately zero.

With economic growth, the SOE can have a permanent current account deficit and thus permanently increasing NFD and yet remain solvent forever. What is in the long run needed for equilibrium, however, is that the foreign debt does not grow faster than GDP. As we saw in Chapter 13, this condition will be satisfied if net exports as a fraction of GDP are sufficient to cover the growth-corrected interest payments on the debt. (This analysis ignores that the scope for writing enforceable international credit contracts is somewhat limited and so, in practice, there is likely to be an upper bound on the debt-income ratio acceptable to the lenders. Such a bound is in fact apt to be operative well before the foreign debt moves beyond the value of the capital stock in the economy.)

Overall effect of an oil price shock

Returning to the model, without economic growth, analyzed in detail above, let us summarize. An oil price shock such that p_M shifts to a higher (constant) level implies a lower equilibrium real wage, $w_t = w(K_t, p_M)$, both on impact and in the longer run. The impact effect comes from lower input of oil, hence a lower marginal productivity of labor, cf. (15.20). This implies, on impact, a fall in H_t, see (15.38), and therefore also in C_t, see (15.37). In addition, as was shown in Section 15.1.2, K_t is gradually reduced over time and this decreases output and the marginal productivity of labor further. As a result the long-run values of H and A become lower than before, and so does the long-run value of C. Whether in the long run net foreign assets, A_f^*, and net exports, NX^*, become lower or not we cannot know, because the fall in national wealth, A^*, may, but need not, be larger than the fall in the capital stock, K^*.

To summarize: The overall effect of an adverse oil price shock is an economic contraction. If the model were extended by including short-term Keynesian demand effects, arising from the purchasing power of consumers being undermined by a sudden increase in the general price level, then the economic contraction may become more severe, leading to a pronounced recession.

Going further outside the model we could imagine that trade unions, by demanding compensation for price increases, resist the real wage decrease required for unchanged employment, when the oil price rises. As a result unemployment tends to go up. If in addition the wage-price spiral is accommodated by monetary policy, as after the first oil price shock in 1973-74, then simultaneous high inflation and low output may arise. This is exactly the phenomenon of *stagflation* that we saw in the aftermath of the first oil price shock.

15.1.5 General aspects of modeling a small open economy

Let us return to the case of a very impatient society ($\rho > r$) and focus on (15.44) and (15.46). If the mortality rate μ is very small, the model predicts that the country asymptotically mortgages, in addition to its physical capital, *all* its human capital. The long-run prospect could be a very low consumption level. The Ramsey model as well as the Barro model with an operative bequest motive, are examples of models with a very low μ since, effectively, they have $\mu = 0$. Hence, a Ramsey-style model for a small open economy (ignoring technical progress) with $\rho > r$ will satisfy the condition (15.43) and entail $A_t \to -H^*$, implying de-cumulation *forever*, that is, $C_t \to 0$, by (15.45. The fact that Ramsey-style models can predict such outcomes, is a warning that such models are in some contexts of limited value.

If, on the other hand, $\rho < r$, then the Ramsey model implies low consumption and high saving. Indeed, the country will forever accumulate financial claims on the rest of the world. This is because, in the Ramsey model the Keynes-Ramsey rule holds not only at the individual level, but also at the aggregate level. Eventually, the country becomes a large economy and begins to affect the world interest rate, contradicting the assumption that it is a small open economy.

To avoid these extreme outcomes, when applying the Ramsey model for studying a small open economy, one has to assume $\rho = r$. But this is an unwelcome knife-edge condition. Recall that a model is said to build on a *knife-edge* condition if the model imposes a particular value on a parameter within a continuum of possible values such that an actual deviation from this value alters the dynamics qualitatively. In that case the dynamics associated with the knife-edge value are not a likely outcome and not representative.

It is otherwise with the Blanchard OLG model, where the generation replacement effect implies that the Keynes-Ramsey rule does *not* hold at the aggregate level. Therefore, the OLG model for a small open economy needs no knife-edge condition on parameters. The model works well whatever the size relation between ρ and r, as long as the stability condition (15.43) is satisfied. Or, to be more precise: the Blanchard model works well in the case $\rho < r$; in the opposite case, where $\rho > r$, the model works at least better than the Ramsey model, because it never implies that $C_t \rightarrow 0$ in the long run.

It should be admitted, however, that in the case of a very impatient country ($\rho > r$), even the OLG model implies a counterfactual prediction. What (15.46) tells us is that the impatient small open economy in a sense asymptotically mortgages all of its physical capital and part of its human capital. The OLG model predicts this will happen, *if* financial markets are perfect, and *if* the political sphere does not intervene. It certainly seems unlikely that an economic development, ending up with negative national wealth, is going to be observed in practice. There are two - complementary - explanations of this.

First, the international credit market is far from perfect. Because a full-scale supranational legal authority comparable with domestic courts is lacking, credit default risk in international lending is generally a more serious problem than in domestic lending. Physical capital can to some extent be used as a collateral on foreign loans, while human wealth is not suitable. Human wealth cannot be repossessed. This implies a constraint on the ability to borrow.[10] And lenders' risk perceptions depend on the level of debt.

Second, long before *all* the physical capital of an impatient country is mortgaged or have directly become owned by foreigners, the government presumably would intervene. In fear of losing national independence, it would use its political power to end the pawning of economic resources to foreigners.

This is a reminder, that we should not forget that the economic sphere of a society is just one side of the society. Politics as well as culture and religion are other sides. The economic outcome may be conditioned on these social factors, and the interaction of all these spheres determines the final outcome.

15.2 Housing market dynamics

(figures not yet updated to comply with $\alpha = 1$ and other recent changes).

The housing market is from a macroeconomic point of view important for several reasons: a) residential investment is typically of magnitude about 5 percent of GDP and roughly a half of total fixed investment; b) housing makes up a weighty part of the consumption budget; c) housing wealth makes up a substantial part of private wealth of a major fraction of the population; d) mortgage debt make up a large part of households' liabilities; and e) house prices and construction

[10] We have been speaking as if domestic residents own the physical capital stock in the country, but have obtained part or all the financing of the stock by issuing bonds to foreigners. The results would not change if we allowed for foreign direct investment. Then foreigners would themselves own part of the physical capital rather than bonds. In such a context a similar constraint on foreign investment is likely to arise, since a foreigner can buy a factory or the shares issued by a firm, but it is difficult to buy someone else's stream of future labour income.

activity fluctuate considerably and are strongly positively correlated with each other and with aggregate output.

The analysis will be based on a simple partial equilibrium model with rising marginal construction costs. The aim is to throw light on short-to-medium-run dynamics.

Let time be continuous. Let H_t denote the aggregate housing stock at time t and S_t the aggregate flow of housing services at time t. Ignoring heterogeneity, the housing *stock* can be measured in terms of m^2 floor area available for accommodation at a given point in time. For convenience we will talk about the stock as a certain number of houses of a standardized size. The supply of housing *services* at time t constitutes a *flow*, thereby being measured *per time unit*, say per year. The two concepts are related through $S_t = u \cdot H_t$, where u is the service flow per year per house. If the service flow is measured in square meter-months, u equals the number of square meters of a "normal-sized" house times 12. Let us define one unit of housing service per year to mean disposal of a house of standard size one year. So, $u = 1$, and we have

$$S_t = 1 \cdot H_t. \tag{15.48}$$

15.2.1 The housing service market and the house market

There are two goods, houses and housing services, and therefore also two markets and two prices:

$$\begin{aligned} p_t &= \text{the (real) price of a "normal-sized'" house at time } t, \\ R_t &= \text{the rental rate } \equiv \text{ the (real) price of housing services at time } t. \end{aligned}$$

The price R_t of housing services is known as the *rental rate* at the housing market. Buying a housing service means *renting* the apartment or the house for a certain period. Or, if we consider an owner-occupied house (or apartment), R_t is the imputed rental rate, that is, the owner's opportunity cost of occupying the house. The prices R_t and p_t are measured in *real* terms, or more precisely, they are deflated by the consumer price index. We assume perfect competition in both markets.

The market for housing services

In the short run the housing stock is historically given. Construction is time-consuming and houses cannot be imported. Owing to the long life of houses, investment in new houses per year tends to be a small proportion of the available housing stock (in advanced economies about 3 percent, say). So also the supply, S_t, of housing services is given in the short run.

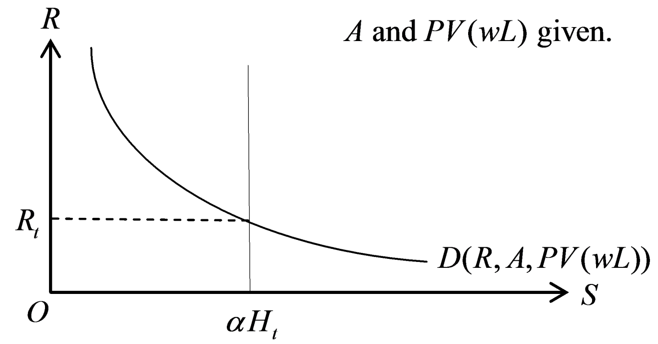

Figure 15.6: Supply and demand in the market for housing services at time t.

Suppose the aggregate demand for housing services at time t is

$$S_t^d = D(R_t)X_t, , \qquad D' < 0, \tag{15.49}$$

where X_t is a composite of factors other than R_t affecting demand. That demand depends negatively on the rental rate reflects that both the substitution effect and the income effect of a higher rental rate are negative. The wealth effect on housing demand of a higher rental rate is likely to be positive for owners and negative for tenants.[11] In our partial equilibrium analysis we take X_t as exogenous. In Section 15.2.3 we need to be more specific concerning X_t and shall identify it with aggregate wealth (financial plus human) in the economy.

The market for housing services at time t is depicted in Fig. 15.6. The supply of housing services is given by H_t. The position of the downward-sloping demand curve, $D(R_t)$, depends on the background factor, X_t. The market clearing rental rate, R_t, is determined by the equilibrium condition

$$D(R_t)X_t = 1 \cdot H_t. \tag{15.50}$$

Because the supply of housing services is inelastic in the short run, R_t immediately moves up or down as the demand curve shifts rightward or leftward, respectively, i.e., as a positive or negative shock to X_t occurs.

[11] A simple microeconomic "rationale" behind the aggregate demand function (15.49) is obtained by assuming an instantaneous utility function $u(h_t, c_t) = \ln(h_t^{\gamma} c_t^{1-\gamma})$, where $0 < \gamma < 1$, and h_t is consumption of housing services at time t, whereas c_t is non-housing consumption. Then the share of housing expenditures in the total instantaneous consumption budget will equal the constant γ. This is broadly in line with empirical evidence for the US (Davis and Heathcote, 2005). In turn, according to standard neoclassical theory, the total consumption budget will be an increasing function of total wealth of the household, cf. Chapter 9. Separation between the two components of wealth, A and $PV(wl)$, is relevant when credit markets are imperfect.

The equilibrium condition (15.50) determines R_t as an implicit function of H_t/X_t :

$$R_t = R(H_t/X_t), \qquad R' = \frac{1}{D'(R(H_t/X_t))} < 0. \tag{15.51}$$

The market for existing houses

Because a house is a durable good with market value, it is an *asset*. This asset typically constitutes a substantial share of the wealth of a large fraction of the population, the house-owners. At the same time the supply of the asset can change only slowly.

Assume there is an exogenous and constant risk-free real interest rate $r > 0$. This is a standard assumption in partial equilibrium analysis. If the economy is a small open economy with perfect capital mobility, the exogeneity of r (if not constancy) is warranted even in general equilibrium analysis.

Considering the asset motive associated with housing, a series of aspects are central. We let houses depreciate physically at a constant rate $\delta > 0$. Suppose there is a constant tax rate $\tau_R \in [0, 1)$ applied to rental income (possibly imputed) after allowance for depreciation. In case of an owner-occupied house the owner must pay the tax $\tau_R(R_t - \delta p_t)$ out of the imputed income $(R_t - \delta p_t)$ per house per year. Assume further there is a constant property tax (real estate tax) $\tau_p \geq 0$ applied to the market value of houses. Finally, suppose that a constant tax rate $\tau_r \in [0, 1)$ applies to interest income. There is symmetry in the sense that if you are a debtor and have negative interest income, then the tax acts as a rebate. We assume capital gains are not taxed and we ignore all complications arising from the fact that most countries have tax systems based on nominal income rather than real income. In a low-inflation world this limitation may not be serious.[12]

Suppose there are no credit market imperfections, no transaction costs, and no uncertainty. Assume further that the user of housing services value these services independently of whether he/she owns or rent. Under these circumstances the price of houses, p_t, will adjust so that the expected after-tax rate of return on owning a house equals the after-tax rate of return on a safe bond. We thus have the no-arbitrage condition

$$\frac{(1 - \tau_R)(R(H_t/X_t) - \delta p_t) - \tau_p p_t + \dot{p}_t^e}{p_t} = (1 - \tau_r)r, \tag{15.52}$$

where $\dot{p}_t^e$ denotes the expected capital gain per time unit (so far $\dot{p}_t^e$ is just a commonly held subjective expectation).

[12] Note, however, that if all capital income should be taxed at the same rate, capital gains should also be taxed at the rate τ_r, and τ_R should equal τ_r.

For given $\dot{p}_t^e$ we find the equilibrium price

$$p_t = \frac{(1-\tau_R)R(H_t/X_t) + \dot{p}_t^e}{(1-\tau_r)r + (1-\tau_R)\delta + \tau_p}.$$

Thus p_t depends on H_t, $\dot{p}_t^e$, r, and tax rates in the following way:

$$\begin{aligned}
\frac{\partial p_t}{\partial H_t} &= \frac{(1-\tau_R)R'(H_t/X_t)}{(1-\tau_r)r + (1-\tau_R)\delta + \tau_p} < 0, \\
\frac{\partial p_t}{\partial \dot{p}_t^e} &= \frac{1}{(1-\tau_r)r + (1-\tau_R)\delta + \tau_p} > 0, \\
\frac{\partial p_t}{\partial \tau_R} &= \frac{-\left[(1-\tau_r)r + \tau_p\right]R(H_t/X_t) + \delta\dot{p}_t^e}{\left[(1-\tau_r)r + (1-\tau_R)\delta + \tau_p\right]^2} \lesseqqgtr 0 \text{ for } \dot{p}_t^e \lesseqqgtr \frac{\left[(1-\tau_r)r + \tau_p\right]R(H_t/X_t)}{\delta}, \\
\frac{\partial p_t}{\partial \tau_p} &= -\frac{(1-\tau_R)R(H_t/X_t) + \dot{p}_t^e}{\left[(1-\tau_r)r + (1-\tau_R)\delta + \tau_p\right]^2} < 0, \\
\frac{\partial p_t}{\partial \tau_r} &= \frac{\left[(1-\tau_R)R(H_t/X_t) + \dot{p}_t^e\right]r}{\left[(1-\tau_r)r + (1-\tau_R)\delta + \tau_p\right]^2} > 0, \\
\frac{\partial p_t}{\partial r} &= -\frac{\left[(1-\tau_R)R(H_t/X_t) + \dot{p}_t^e\right](1-\tau_r)}{\left[(1-\tau_r)r + (1-\tau_R)\delta + \tau_p\right]^2} < 0,
\end{aligned}$$

where the sign of the last three derivatives are conditional on $\dot{p}_t^e$ being nonnegative or at least not "too negative".

Note that a higher expected increase in p_t, $\dot{p}_t^e$, implies a higher house price p_t. Over time this feeds back and may confirm and sustain the expectation, thus generating a further rise in p_t. Like other assets, a house is thus a good with the property that the expectation of price increases makes buying more attractive from an investment point of view and may become self-fulfilling if the expectation is generally held.

15.2.2 Residential construction

It takes time for the stock H_t to change. While manufacturing typically involves mass production of similar items, construction is generally done on location for a known client and within intricate legal requirements. It is time-consuming to design, contract, and execute the sequential steps involved in residential construction. Congestion and bottlenecks may easily arise. Careful guidance, supervision, and monitoring is needed. These features give rise to fixed costs (to management, architects etc.) and thereby rising marginal costs in the short run.

The construction process

Assume the construction industry is competitive. At time t the *representative construction firm* produces B_t units of housing per time unit (B for "building"), thereby increasing the aggregate housing stock according to

$$\dot{H}_t = B_t - \delta H_t, \qquad \delta > 0. \tag{15.53}$$

The construction technology is described the following way:

$$B_t = \tilde{F}(K_t, A_t L_t, E_t \bar{M}) \equiv \bar{F}(F(\underbrace{K_t, A_t L_t}_{CRS}), E_t \bar{M}) = \bar{F}(\underset{DRS}{I_t}, E_t \bar{M}) \equiv T(\underset{DRS}{I_t}, E_t), \tag{15.54}$$

where the function $\tilde{F}$ is increasing in its three arguments, K_t is input of capital, $A_t L_t$ is blue-collar labor in efficiency units, L_t being blue-collar labor in hours, $E_t \bar{M}$ is management labor in efficiency units, $\bar{M}$ being management in hours. The latter includes hours of scarce specialists like architects, engineers, and lawyers. Capital and blue-collar labor are considered variable production factors even in the short run. In contrast, it is costly and takes time and effort to change managerial capacity. Hence, we treat $\bar{M}$ as a fixed production factor in the short run and as only changing slowly over time. In a short-to-medium run perspective, $\bar{M}$ is thus close to being time-independent which we, to help intuition, indicate by omitting the subscript t.

Capital and blue-collar labor produces components for residential construction − intermediate goods − in the amount $I_t = F(K_t, A_t L_t)$ per time unit; F is a CRS production function and is "nested" in the "global" production function, $\bar{F}$. Construction is thus modeled as if it makes up a two-stage process. First, capital and blue-collar labor produce intermediate goods for construction. Next, management accomplishes quality checks and "assembling" of these intermediate goods into new houses or at least into final new components built into existing houses. The final housing output is measured in units corresponding to a "standard house". This does not rule out that a considerable part of the output is really in the form of renovations, additions of a room etc.

In view of F featuring CRS and K_t and L being variable production factors even in the short run, intermediate goods are produced on a routine basis at constant unit costs. We let this cost per unit of I_t be denoted c in real terms. In our short-to-medium run perspective we treat c as time-independent. The efficiency factor A_t mirrors the "economy-wide" technology level, growing at the general rate of technical progress in the economy, g. In contrast, E_t measures sector-specific efficiency which may reflect accumulated learning in the construction industry.

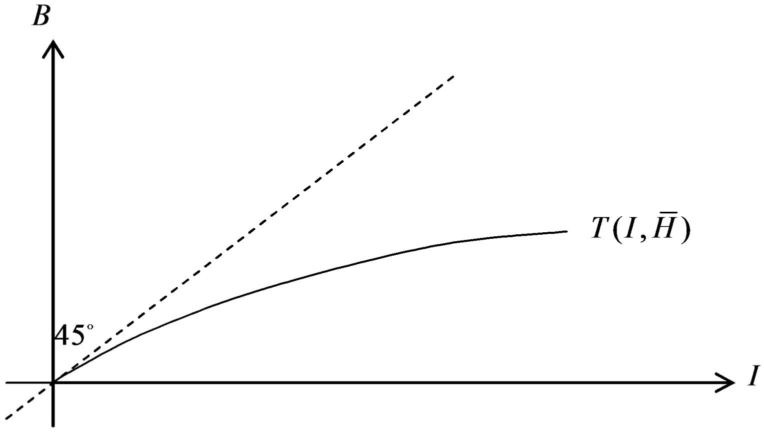

Figure 15.7: The number of new houses as a function of residential investment (for given $E = \bar{H}$).

Finally, at the end of the chain in (15.54), to simplify notation, we suppress the "almost constant" $\bar{M}$ and introduce the "transformation function" $B_t = T(I_t, E_t)$. This function has decreasing returns with respect to I_t : the larger is I_t, the smaller is the rate at which a unit increase in I_t is transformed into new houses, as illustrated in Fig. 15.7. The interpretation is that more construction activity per time unit means that a larger fraction of I is "wasted" because of control, coordination, and communication difficulties due to scarce managerial capacity. An alternative or supplementary cause behind the decreasing marginal productivity of I_t is that negative externalities may arise due to congestion at building sites in a construction boom.

The second argument in the transformation function is the construction efficiency level E_t. Higher efficiency means that the intermediate goods can be designed in a better way and be "assembled" in a more efficient way. This results in higher productivity of a given I than otherwise, hence $T_E > 0$.

The following list summarizes the implied properties of the transformation function:

$$T(0, E) = 0, \quad T_I(0, E) = 1, \quad T_I > 0, \quad T_{II} < 0, \quad T_E > 0. \qquad (15.55)$$

The properties $T_I(0, E) = 1$ and $T_{II} < 0$ imply $T_I(I, E) < 1$ for $I > 0$, as visualized in Fig. 15.7.

Note that until Section 15.2.3, we do not require the "global" production function, whether in the form of $\tilde{F}$, $\bar{F}$, or T, to have CRS rather than DRS within the relevant range of output. An example satisfying all the conditions in (15.55) and *also* the condition of CRS of the "global" production function is a

CES function with elasticity of substitution < 1:[13]

$$T(I,E) = \alpha^{-1/\beta}(aI^{\beta} + (1-a)E^{\beta})^{1/\beta}, \quad \text{with } 0 < \alpha < 1, \text{ and } \beta < 0.$$

Remark. From the perspective of Tobin's q-theory of investment, we might let the "waste" be represented by a kind of adjustment cost function $G(I,E)$ akin to that considered in Chapter 14. Then $T(I,E) \equiv I - G(I,E)$. In Chapter 14 convex adjustment costs were associated with the installation of firms' fixed capital. Our main focus was on the case where the adjustment costs acts as a reduction in the firms' output available for sale. In construction we may speak of comparable costs acting as a reduction in the productivity of the intermediate goods in the construction process. It is easily seen that, on the one hand, all the properties of G *required* in Chapter 14.1.1 when $I \geq 0$ are maintained. On the other hand, not all properties required of T in (15.55) need be satisfied in Tobin's q-theory (see Appendix B). □

Profit maximization

Here we temporarily skip the explicit dating of time-dependent variables. The representative construction firm takes the current efficiency level, E, as given. The gross revenue of the firm is pB and variable costs are cI. At any instant, given the market price p, the firm maximizes profit (in the sense of revenue minus variable costs):

$$\begin{aligned} \max_I \Pi &= pB - cI \qquad \text{s.t.} \qquad B = T(I,E) \text{ and} \\ I &\geq 0. \end{aligned}$$

Inserting $B = T(I,E)$, we find that an interior solution satisfies

$$\frac{d\Pi}{dI} = pT_I(I,E) - c = 0, \text{ i.e., } \frac{p}{c}T_I(I,E) = 1. \tag{15.56}$$

In view of $T_I(I,E) < 1$ for $I > 0$, the latter equation has a solution $I > 0$ only if $p > c$. For $p \leq c$, we get the corner solution $I = 0$. Naturally, when the current market price of houses is below marginal construction cost (which equals $c/(T_I(I,E) \geq c)$, no new houses will be built.[14] This is a desired property of the model since sometimes in the real world, residential construction comes to a

[13] As shown in the appendix to Chapter 4, by defining $T(I,E) = 0$ when $I = 0$ or $E = 0$, the domain of the CES function can be extended to include all $(I,E) \in \mathbb{R}^2_{++}$ also when $\beta < 0$, while maintaining continuity.

[14] How to come from the transformation function $T(I,E)$ to the marginal cost schedule is detailed in Appendix C.

standstill. On the other hand, when $p > c$, the construction firm will supply new houses up to the point where the rising marginal cost equals the current house price, p.

A characterization of the optimal flow of intermediates, I, is obtained the following way. Since T_{II}, the first-order condition (15.56) defines, for $p > c$, construction activity, I, as an implicit function of p/c and E :

$$I = \Phi\left(p/c, E\right), \qquad \text{where } \Phi(1, E) = 0. \tag{15.57}$$

After substituting (15.57) into (15.56), by implicit differentiation with respect to p/c in (15.56) and rearranging, we find the partial derivative

$$\Phi_{p/c} = \frac{\partial I}{\partial(p/c)} = \frac{-1}{(p/c)^2 T_{II}(I, E)} > 0,$$

where the argument I can be written as in (15.57).

The CRS case

If the global production function $\bar{F}$ is homogeneous of degree one, so is the transformation function T. In that case we have $B = T(I/E, 1)E$. Moreover, by Euler's theorem, $T_I(I, E)$ is homogeneous of degree 0. So, the first-order condition (15.56) can be written

$$\frac{p}{c} T_I\left(\frac{I}{E}, 1\right) = 1. \tag{15.58}$$

This equation defines input of efficiency-corrected intermediates, I/E, as an implicit function of p_t/c :

$$\frac{I}{E} = \varphi\left(\frac{p}{c}\right), \qquad \text{where } \varphi(1) = 0. \tag{15.59}$$

By implicit differentiation with respect to p/c in the first-order condition (15.58) and rearranging, we find

$$\varphi' = \frac{-1}{(p/c)^2 T_{II}\left(I/E, 1\right)} > 0,$$

where I/E from (15.59) can be inserted. A construction activity function φ with this property is shown in Fig. 15.8, where $c = 1$. (NB: the figure not yet adjusted to recent changes in model).

Remark. Like Tobin's q, the house price p is the market value of a produced asset whose supply changes only slowly. As is the case for firms' fixed capital, there are

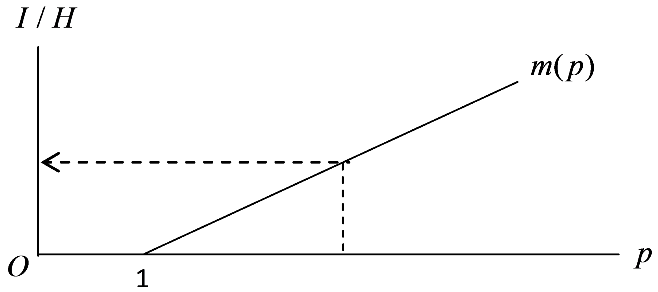

Figure 15.8: Construction activity (relative to the housing stock) as a function of the market price of houses ($c = 1$).

strictly convex stock adjustment costs, represented by the rising marginal construction costs. As a result the stock of houses does not change instantaneously if for instance p changes. But as shown by the above analysis, the flow variable, residential construction, responds to p in a way similar to the way firm's fixed-capital investment responds to Tobin's q according to the q theory. Tobin's q is defined as the economy-wide ratio $V/(p_I K)$, where V is the market value of the firms, p_I is a price index for investment goods, and K is the stock of physical capital. The analogue ratio in the housing sector is $V^{(H)}/(p_I \cdot H) \equiv p \cdot H/(p_I \cdot H)$ $= p/c$, in view of $p_I = c$. A higher p/c results in more construction activity. □

15.2.3 Dynamics under perfect foresight

In our dynamic analysis, we concentrate on the case where $\bar{F}$ is homogeneous of degree one. Since the focus is on the evolution over time, we allow for slow growth in management labor, M_t, namely at the rate of population growth $n \geq 0$. So (15.54) is replaced by

$$B_t = \tilde{F}(\underbrace{K_t, A_t L_t, E_t M_t}_{CRS}) \equiv \bar{F}(F(\underbrace{K_t, A_t L_t}_{CRS}), E_t M_t) = \bar{F}(\underset{DRS}{I_t}, E_t M_t),$$

where $M_t = M_0 e^{nt}$. Let b_t denote the flow of new houses relative to management measured in efficiency units. So

$$b_t \equiv \frac{B_t}{E_t M_t} = \frac{\bar{F}(I_t, E_t M_t)}{E_t M_t} = \bar{F}\left(\frac{I_t}{E_t M_t}, 1\right) = \bar{F}\left(\varphi\left(\frac{p_t}{c}\right), 1\right) \equiv b\left(\frac{p_t}{c}\right), \quad (15.60)$$

where $b(1) = \bar{F}(\varphi(1), 1) = \bar{F}(0, 1) = 0$, $b' = \bar{F}_I \varphi' > 0$.

In view of (15.60), we have from (15.53) that

$$\dot{H}_t = B_t - \delta H_t = b\left(\frac{p_t}{c}\right) E_t M_t - \delta H_t. \quad (15.61)$$

Assuming rational expectations in our model without stochastic elements is equivalent to assuming *perfect foresight*, that is, $\dot{p}_t^e = \dot{p}_t$ for all t. Then we can write the no-arbitrage condition (15.52) as an ordinary first-order differential equation:

$$\dot{p}_t = [(1-\tau_r)r + (1-\tau_R)\delta + \tau_p]\, p_t - (1-\tau_R)R(H_t/X_t), \tag{15.62}$$

where $R' < 0$.

The two coupled differential equations, (15.61) and (15.62), make up a dynamic system in H_t and p_t. The system is not autonomous, however, because it contains time-dependent exogenous variables, X_t and $E_t M_t$.

As already mentioned, we assume that M_t grows at the rate of population growth $n \geq 0$. The construction efficiency, E, is likely to rise smoothly over time. We will simply assume its growth rate, g_E, is a positive constant. So $E_t M_t$ grows at the rate $g_E + n > 0$.

Even regarding the demand factor X_t we will, in our partial equilibrium perspective, ignore business cycle fluctuations and simply assume that it for all t equals the trend level of aggregate total wealth (financial plus human). In turn, this wealth is naturally assumed to grow at the rate $g+n > 0$, i.e., the sum of the general rate of technical progress in the economy, g, and the rate of population growth, n.

There are now three cases to consider, depending on whether $g_E = g$, $g_E < g$, or $g_E > g$.

Case 1: $g_E = g$. Here both X_t and $E_t M_t$ grow at the rate $g + n$. By a proper choice of measurement units we can then obtain $E_t M_t = X_t$ for all t. Let the "trend-corrected housing stock" be defined as $\tilde{H}_t \equiv H_t/X_t$. Then $\dot{\tilde{H}}_t/\tilde{H}_t = \dot{H}_t/H_t - \dot{X}_t/X_t$ and thereby $\dot{\tilde{H}}_t = (\dot{H}_t/H_t - (g+n))\tilde{H}_t$. Into this we substitute (15.61) and get

$$\dot{\tilde{H}}_t = b\left(\frac{p_t}{c}\right) - (\delta + g + n)\tilde{H}_t. \tag{15.63}$$

In combination with

$$\dot{p}_t = [(1-\tau_r)r + (1-\tau_R)\delta + \tau_p]\, p_t - (1-\tau_R)R(\tilde{H}_t), \tag{15.64}$$

this makes up an autonomous dynamic system in $\tilde{H}$ and p.

The corresponding phase diagram is shown in Fig. 15.9. (NB: the figure not yet adjusted to recent changes in model).We have $\dot{\tilde{H}} = 0$ for $b(p/c) = (\delta+g+n)\tilde{H}$. In view of $b' > 0$, this $\dot{\tilde{H}} = 0$ locus is an upward-sloping curve in the diagram. In view of $b(1) = 0$, the curve intersects the ordinate axis at the ordinate c. The direction of movement of H is positive above the curve and negative below.

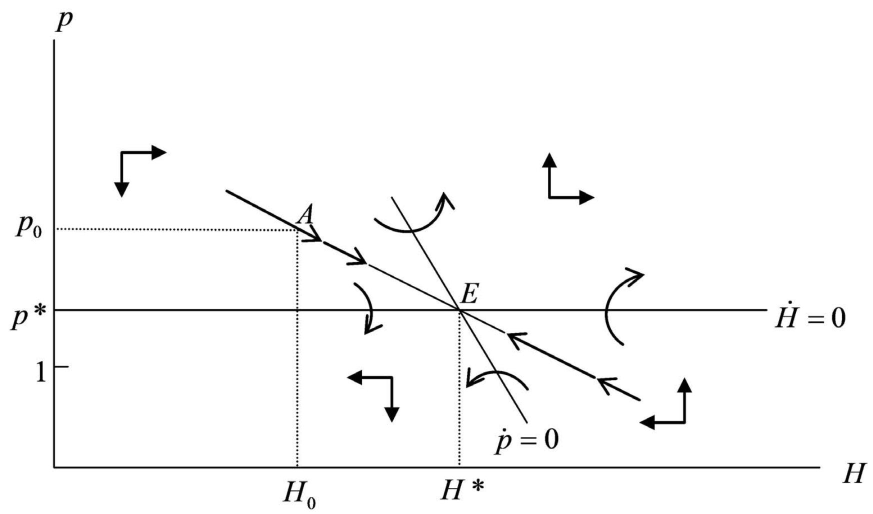

Figure 15.9: Phase diagram for the dynamics of construction and house prices ($c = 1$, $T(I, E)$ homogeneous of degree 1, $g_E = g$).

We have $\dot{p} = 0$ for $p = (1 - \tau_R)R(\tilde{H})/\left[(1 - \tau_r)r + (1 - \tau_R)\delta + \tau_p\right]$. Since $R'(\tilde{H}) < 0$, the $\dot{p} = 0$ locus in the diagram has negative slope. There is a unique steady state, the point E with coordinates $\tilde{H}^*$ and p^*. To the right of the $\dot{p} = 0$ locus, p is rising, and to the left p is falling. The directions of movement of $\tilde{H}$ and p in the different regions of the phase plane are indicated by the arrows in the figure. The arrows show that the steady state is a saddle point. The initial housing stock, $\tilde{H}_0$, is predetermined. Hence, at time $t = 0$, the economic system must be somewhere on the vertical line $\tilde{H} = \tilde{H}_0$.

The question now is whether there can be asset price bubbles in the system. An *asset price bubble* is present if the market value of the asset for some time systematically exceeds its *fundamental value,* which we may call $\hat{p}_t$. This is the present discounted value of the expected future services or dividends from the asset.[15] The divergent trajectories ultimately moving North-East in the phase diagram are, by construction, bubbly price paths consistent with the re-written no-arbitrage condition (15.64). They are thus candidates for asset price bubbles generated by self-fulfilling expectations. Such explosive price paths can hardly be realized, however, given the assumption of rational expectations, here perfect foresight. The argument is given in Section 15.2.4 below.

As also a *negative* bubble is implausible (corresponding to the divergent trajectories ultimately moving South-West in the phase diagram), we are left with the converging path as the unique solution to the model. At time 0 the residential

[15] For details, see Appendix D.

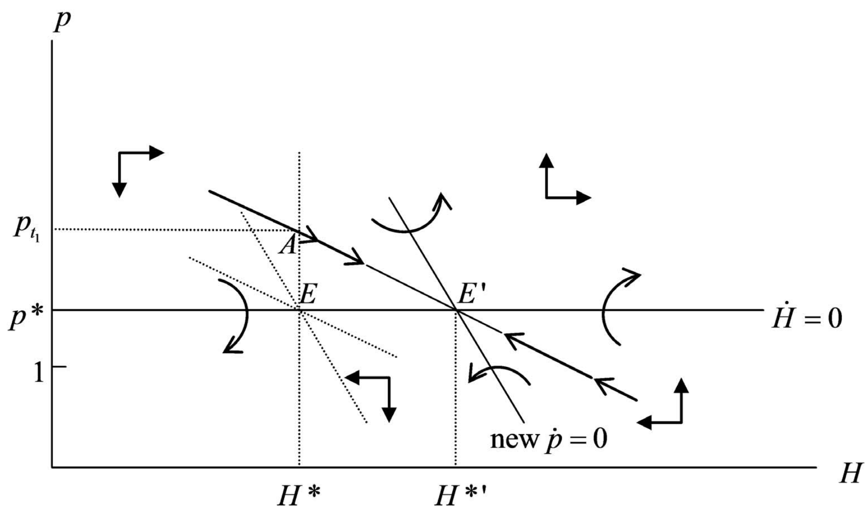

Figure 15.10: Response to a fall in the property tax ($c = 1$, $T(I, E)$ homogeneous of degree 1, $g_E = g$).

construction sector will be at the point A in the diagram and then it will move along the saddle path. After some time the housing stock and the house price settle down at the steady state, E.

Case 2: $g_E < g$. This case seems to characterize the US in recent decades (Iacoviello and Neri, 2010) as well as Denmark (DØR).

(NB: the remainder of the chapter not adjusted to recent changes in model)

Effect of a fall in the property tax

In Denmark, in the early 2000s, the government replaced the rental value tax, τ_R, on owner-occupied houses by a lift in the property tax, τ_p, combined with a *nominal* "tax freeze", implying that τ_p has been gradually decreasing in real terms in view of inflation. Indeed, if T is the property tax in real terms, we have $T = \tau_p p$, and in nominal terms $TP = \tau_p pP \equiv \tau_p P_H$, where P is the nominal price level in the economy and P_H is the nominal price of a "standard house". We see that constancy of TP requires τ_p decreasing if P_H is increasing. Hence, let us study the effect on the housing market of a fall in τ_p.

Suppose the residential construction sector has been in the steady state E in Fig. 15.10 until time t_1. (NB: the figure not yet adjusted to recent changes in model). Then there is an unanticipated downward shift in the property tax τ_p to a new constant level τ_p' rightly expected to last forever in the future. The resulting evolution of the system is shown in the figure. The new steady state is called E'. The new medium-run level of H is $H^{*\prime} > H^*$, because $R'(H) < 0$. On impact, house owners benefit from a capital gain in that p jumps up to the point where the vertical line $H = H^*$ crosses the new (downward-sloping) saddle path. The intuition is that the after-tax return on owning a house has been increased. Hence, by arbitrage the market price p rises to a level such that the after-tax rate of return on houses is as before, namely equal to $(1 - \tau_r)r$. After t_1, owing to the high p relative to the unchanged building cost schedule, H increases gradually and p falls gradually (due to R falling in response to the rising H). This continues until the new steady state is reached with unchanged p^*, but higher H.

The dichotomy between the short and the medium run

There is a dichotomy between the price and quantity adjustment in the short and medium run:

1. In the *short run*, H, hence also the supply of housing services, is given. The rental rate R as well as the house price p immediately shifts up (down) if the demand for housing services shifts up (down).

2. In the *medium run* (i.e., without new disturbances), it is H that adjusts and does so gradually. The adjustment of H is in a direction indicated by the sign of the initial price difference, $p - p^*$, which in turn reflects the initial position of the demand curve in Fig. 15.6. On the other hand, the house price, p, converges toward the cost-determined level, p^*. This price level is constant as long as technical progress in the production of intermediate goods for construction follows the general trend in the economy.

15.2.4 Discussion

In many countries a part of the housing market is under some kind of rent control. Then there is, of course, rationing on the demand side of the housing market. It may still be possible to use the model in a modified version since the part of the housing market, which is *not* under regulation and therefore has a market determined price, p, usually includes the new building activity.

We have carried out partial equilibrium analysis in a simplified framework. Possible refinements of the analysis include considering household optimization with an explicit distinction between durable consumption (housing demand) and non-durable consumption and allowing uncertainty and credit market imperfections. Allowing for convex capital adjustment costs in the production of the intermediate construction goods would reinforce the tendency to rising marginal costs, but also noticeably complicate the model by adding an extra state variable with associated shadow price. A more complete analysis would also include land prices and ground rent.

Finally, a general equilibrium approach would take into account the feedbacks on aggregate employment and perceived aggregate wealth from changes in construction activity and p. Feedbacks on aggregate financial wealth from changes in p are more intricate than one might imagine at first glance. At least in a representative agent model everybody is an average citizen and owns the house she lives in. Nobody is better off by a rise in house prices. In a model with heterogeneous agents, those who own more houses than they use themselves gain by a rise in house prices. And those in the opposite situation lose. Whether and how aggregate consumption is affected depends on differences in the marginal propensity to consume and on institutional circumstances concerning collateral in credit markets. In two papers by Case, Quigley, and Shiller (2005, 2011) empirical evidence of a positive relationship between consumption and housing wealth in the US is furnished.

The issue of housing bubbles After a decade of sharply rising house prices, the US experienced between 2006 and 2009 a fall in house prices of about 30% (Shiller,), in Denmark about 20% (Economic Council, Fall 2011). In Section 15.2.3 we argued briefly that in the present model with rational (model consistent) expectations, housing bubbles can be ruled out. Let us here go a little more into detail about the concepts involved.

The question is whether the large empirical volatility in house prices should be seen as reflecting the rise and burst of housing bubbles or just volatility of fundamentals. An *asset price bubble* is present if the market price, p_t, of the asset exceeds the *fundamental value* $\hat{p}_t$. The latter is the present value of the expected future services or dividends from the asset (for instance a house) and can be found as the solution to the differential equation (15.64), assuming absence of asset price bubbles (see Appendix D).

A *rational* asset price bubble is an asset price bubble that is consistent with the no-arbitrage condition for the asset, here (15.52), when agents have rational expectations. In the absence of stochastic elements in our model, rational expectations amounts to perfect foresight. In Section 15.2.3 we claimed that rational bubbles are inconsistent with the present model. Let us briefly see why.

Relative to a hypothetical market price $p_t > \hat{p}_t$, the demand side in the housing market is likely to have a cheaper way of acquiring a house or at least the services of a house. An owner occupying her own house because of its "use value" would prefer to sell it and then rent, paying the rental rate per time unit. The present value of the rent payment stream is $\hat{p}_t$, which is less than the hypothetical market price p_t. And speculators that buy houses with a view to sell later at a substantial capital gain due to a bubbly price path might find it difficult to find buyers. For instance, in a general equilibrium framework it is likely that the large profits obtained due to the bubble would attract entrants to the construction industry and expand capacity so as to reduce not only the rental rate but also production costs, thereby being able to undercut house owners trying to sell at p_t. By backward induction we conclude that under these circumstances, a bubbly price path will not arise in the first place.

Could there be a *negative* bubble, that is, a market price $p_t < \hat{p}_t$? No, in case $p_t < \hat{p}_t$, there will be market participants around eager to buy at a higher price $p'_t \in (p_t, \hat{p}_t)$. The buyer could then either let the house to somebody else, thereby receiving a stream of rental payments of present value equal to $\hat{p}_t$, or the buyer could use the house as home for herself, thereby avoiding to pay the market rents with present value equal to $\hat{p}_t$.

The situation is not essentially different if we add calculable uncertainty to the model. Then we might think of *stochastic housing bubbles*, but can rule them out by similar arguments, now in terms of expected values, as in the deterministic case.[16]

But many economic situations are marked by *fundamental uncertainty.* Then objective expected values do not exist, and fundamental values, bubbles, and rational expectations are not well-defined. This is where the *behavioral finance* literature enters the scene. In that literature speculative bubbles are linked to *market psychology* (herding, fads, etc.). We postpone further discussion of asset price bubbles to Part VI.

15.3 Literature notes

(incomplete)

Poterba (1984).

Attanasio et al., 2009.

Buiter, Housing wealth isn't wealth, WP, London School of Economics, 20-07-2008.

The question of systematic bias in homebuyer's expectations in four U.S. metropolitan areas over the period 2003-2012 is studied in Case, Shiller, and Thompson (2012), based on questionnaire surveys. See also Cheng, Raina, and Xiong (2012).

Shiller (2003) gives an introduction behavioral finance theory.

Campbell and Cocco, 2007.

[16]Including land and unique building sites with specific amenity values into the model will make the argument against rational bubbles less compelling, however (see, e.g., Kocherlakota, 2011).

Himmelberg et al. (2005) survey issues arising when trying to empirically assess the state of house prices: can fundamentals explain the high demand or is existence of a bubble more likely?

Kiyotaki et al. (2011) study the issue of winners and losers in housing markets on the basis of a quantitatively oriented life-cycle model.

Mayer (2011) surveys theory and empirics about bubbles and the cyclical movement of house prices.

Iacoviello and Neri (2010) find .

The phenomenon that fast expansion may reduce efficiency when managerial capability is a fixed production factor is known as a *Penrose effect*, so named after a book from 1959 on management by the American economist Edith Penrose (1914-1996). Uzawa (1969) explores Penrose's ideas in different economic contexts. The construction process is sensitive to managerial capability which is a scarce resource in a construction boom.

15.4 Appendix

A. Complementary inputs (Section 15.1)

In Section 15.1.2 we claimed, without proof, certain properties of the oil demand function and the marginal productivities of capital and labor, respectively, in general equilibrium, given firms' profit maximization subject to a three-factor production function with inputs that exhibit direct complementarity. Here, we use the attributes of the production function F, including (15.2), and the first-order conditions of the representative firm, to derive the claimed signs of the partial derivatives of the functions $M(K, p_M)$, $w(K, p_M)$, and $MPK(K, p_M)$.

First, taking differentials with respect to K and M on both sides of (15.13) gives

$$F_{MK}dK + F_{MM}dM = dp_M.$$

Hence, $\partial M/\partial K = -F_{MK}/F_{MM} > 0$, and $\partial M/\partial p_M = 1/F_{MM} < 0$.

Second, taking differentials with respect to K and p_M on both sides of (15.12) yields

$$dw = F_{LK}dK + F_{LM}(M_K dK + M_{p_M} dp_M).$$

Hence, $\partial w/\partial K = F_{LK} + F_{LM}M_K > 0$, and $\partial w/\partial p_M = F_{LM}M_{p_M} < 0$.

Third, $\partial MPK/\partial p_M = F_{KM}M_{p_M} < 0$, since $F_{KM} > 0$ and $M_{p_M} < 0$. As to the sign of $\partial MPK/\partial K$, observe that

$$\begin{aligned} \partial MPK/\partial K &= F_{KK} + F_{KM}M_K = F_{KK} + F_{KM}(-F_{MK}/F_{MM}) \\ &= \frac{1}{F_{MM}}(F_{KK}F_{MM} - F_{KM}{}^2) < 0, \end{aligned}$$

where the inequality follows from $F_{MM} < 0$, if $F_{KK}F_{MM} - F_{KM}{}^2 > 0$. And the latter inequality does indeed hold. This follows from (15.65) in the lemma below.

Lemma. Let $f(x_1, x_2, x_3)$ be some arbitrary concave C^2-function defined on R^3_+. Assume $f_{ii} < 0$ for $i = 1, 2, 3$, and $f_{ij} > 0, i \neq j$. Then, concavity of f implies that

$$f_{ii}f_{jj} - f_{ij}{}^2 > 0 \quad \text{for } i \neq j. \tag{15.65}$$

Proof. By the general theorem on concave C^2-functions (see Math Tools), f satisfies

$$f_{11} \leq 0, \qquad f_{11}f_{22} - f_{12}{}^2 \geq 0 \text{ and}$$

$$f_{11}(f_{22}f_{33} - f_{23}{}^2) - f_{12}(f_{21}f_{33} - f_{23}f_{31}) + f_{13}(f_{21}f_{32} - f_{22}f_{31}) \leq 0 \tag{15.66}$$

in the interior of R^3_+. Combined with the stated assumptions on f, (15.66) implies (15.65) with $i = 2, j = 3$. In view of symmetry, the numbering of the arguments of f is arbitrary. So (15.65) also holds with $i = 1, j = 3$ as well as $i = 1, j = 2$. □

The lemma applies because F satisfies all the conditions imposed on f in the lemma. First, the direct complementarity condition $f_{ij} > 0, i \neq j$, is directly assumed in (15.2). Second, the condition $f_{ii} < 0$ for $i = 1, 2, 3$ is satisfied by F since, in view of F being neoclassical, the marginal productivities of F are diminishing. Finally, as F in addition to being neoclassical has non-increasing returns to scale, F is concave.

B. The transformation function and the adjustment cost function in Tobin's q-theory (Section 15.2.2)

As mentioned in Section 15.2.2 we may formulate the strictly concave transformation function $T(I, E)$ as being equal to $I - G(I, E)$, where the "waste" is represented by an adjustment cost function $G(I, E)$ familiar from Chapter 14. Then, on the one hand, all the properties of G *required* in Chapter 14.1 when $I \geq 0$ are maintained. On the other hand, not all properties required of T in (15.55) need be satisfied in Tobin's q-theory.

As to the first claim, note that when the function $T(I, H) \equiv I - G(I, H)$ has all the properties stated in (15.55), then the function G must, for $(I, E) \in R^2_{++}$, satisfy:

$$\begin{aligned} G(I,E) &\equiv I - T(I,E), \\ G(0,E) &= 0 - T(0,E) = 0, \\ G_I(I,E) &= 1 - T_I(I,E) \geq 0, \text{ with } G_I \geq 0 \text{ for } I \geq 0, \text{ respectively,} \\ G_{II}(I,E) &= -T_{II}(I,E) > 0 \text{ for all } I \geq 0, \\ G_E(I,E) &= -T_E(I,E) \leq 0, \end{aligned}$$

where the second line is implied by $T_I(0,E) = 1$ and $T_{II} < 0$. These conditions on G are exactly those required in Chapter 14.1.

As to the second claim, a requirement on the function T in (15.55) is that $T_I(0,E) = 1$ and $T_I(I,E) > 0$ for all $I \geq 0$ at the same time as $T_{II} < 0$. This requires that $0 < T_I(I,E) < 1$ for all $I > 0$. For $G(I,E) = I - T(I,E)$ to be consistent with this, we need that $0 < G_I < 1$ for all $I > 0$. So the G function should not be "too convex" in I. We would have to impose the condition that $\lim_{I\to\infty} G_{II} = 0$ holds with "sufficient speed of convergence". Whereas for instance

$$G(I,E) = I - \alpha^{-1/\beta}(\alpha I^\beta + (1-\alpha)E^\beta)^{1/\beta}, \quad \text{with } 0 < \alpha < 1, \text{ and } \beta < 0,$$

will do, a function like $G(I/E) = (\alpha/2)I^2/E$, $\alpha > 0$, will *not* do for large I. Nevertheless, the latter function satisfies all conditions required in Tobin's q-theory as described in Chapter 14. If one would like to use such a quadratic function to represent waste in construction, one could relax the in (15.55) required condition $T_I(I,E) > 0$ to hold only for I below some upper bound.

Finally, we observe that when $T(I,E) \equiv I - G(I,E)$, then, if the function G is homogeneous of degree k, so is the function T, and vice versa.

C. Marginal costs in construction (Section 15.2.2)

We may look at the construction activity of the representative construction firm from the point of view of increasing marginal costs. First, let TC denote the total costs per time unit of the representative construction firm. We have $TC = \bar{f} + TVC$, where $\bar{f}$ is the fixed cost to management and TVC is the total variable cost associated with the construction of B $(= T(I,H))$ new houses per time unit, given the economy-wide stock H. All these costs are measured in real terms. We have $TVC = cI$. The input of intermediates, I, required for building B new houses per time unit is an increasing function of B. Indeed, the equation

$$B = T(I,E), \tag{*}$$

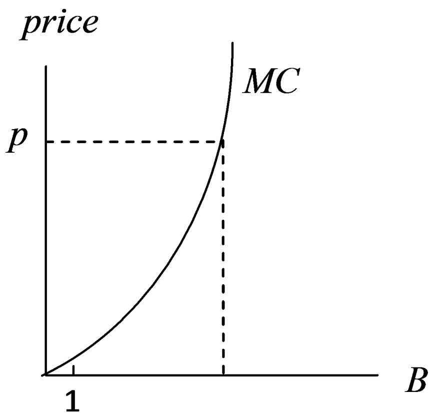

Figure 15.11: Marginal costs in house construction (housing stock given).

where $T_I > 0$, defines I as an implicit function of B and E, say $I = \psi(B, E)$. By implicit differentiation in (*), we find

$$\psi_B = \partial I/\partial B = 1/T_I(\psi(B, E), E) > 1, \quad \text{when } I > 0.$$

So $TVC = cI = c\psi(B, E)$, and short-run marginal cost is

$$MC(c, B, E) = \frac{\partial TVC}{\partial B} = c\psi_B = \frac{c}{T_I(\psi(B, E), E)} > c, \quad \text{when } I > 0. \qquad (**)$$

CLAIM
(i) The short-run marginal cost, MC, of the representative construction firm is increasing in B.

(ii) The construction sector produces new houses up the point where $MC = p$.
(iii) The cost of building *one* new house per time unit is approximately c.

Proof. (i) By (**) and (*),

$$\frac{\partial MC}{\partial B} = \frac{-cT_{II}(\psi(B, E), E)\psi_B}{T_I(\psi(B, E), E)^2} = \frac{-cT_{II}(\psi(B, E), E)}{T_I(\psi(B, E), E)^3} > 0,$$

since $T_I > 0$ and $T_{II} < 0$. (ii) Follows from (**) and the first-order condition (15.56) found in the text. (iii) The cost of building ΔB, when $B = 0$, is $MC(c, \Delta B, E) \approx [c/T_I(0, E)] \cdot \Delta B = c\Delta B = c$ when $\Delta B = 1$, where we have used (**). □

That it is profitable to produce new houses up the point where $MC = p$ is illustrated in Fig. 15.11.

The text assumes that $T_I > 0$ for all $I \geq 0$, hence that the MC curve never becomes vertical. Alternatively, one could assume that at some large level of the flow B, it is impossible to increase B further. This corresponds to the upper section of the MC curve being vertical. In this situation an absolute capacity limit is reached, which in Fig. 15.7 would reflect that for large I, the $T(I, E)$ curve is horizontal. This situation could be the result of the market price p containing an asset price bubble driven by self-reinforcing expectations, thereby spurring a roaring construction boom.

D. Solving the no-arbitrage equation for p_t (Section 15.2.4)

By definition, if there are no house price bubbles, the market price of a house equals its *fundamental value*, i.e., the present value of expected (possibly imputed) after-tax rental income from owning the house. Denoting the fundamental value $\hat{p}_t$, we thus have

$$\begin{aligned}\hat{p}_t &= (1-\tau_R)\int_t^\infty R(\tilde{H}_s)e^{-(\tau_p+\delta)(s-t)}e^{\tau_R\delta(s-t)}e^{-(1-\tau_r)r(s-t)}ds, \qquad (15.67)\\ &= (1-\tau_R)\int_t^\infty R(\tilde{H}_s)e^{-[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p](s-t)}ds,\end{aligned}$$

where the three discount rates appearing in the first line are, first, $\tau_p + \delta$, which reflects the rate of "leakage" from the investment in the house due to the property tax and wear and tear, second, $\tau_R\delta$, which reflects the tax allowance due to wear and tear, and, finally, $(1-\tau_r)r$, which is the usual opportunity cost discount. In the second row we have done an addition of the three discount rates so as to have just one discount factor easily comparable to a key coefficient appearing in the linear differential equation (15.68) below.

In Section 15.2.4 we claimed that in the absence of housing bubbles, the linear differential equation, (15.64), implied by the no-arbitrage equation (15.52) under perfect foresight, has a solution p_t equal to the fundamental value of the house, i.e., $p_t = \hat{p}_t$. To prove this, we write (15.64) on the standard form for a linear differential equation,

$$\dot{p}_t + ap_t = -(1-\tau_R)R(\tilde{H}_t), \qquad (15.68)$$

where

$$a \equiv -\left[(1-\tau_r)r + (1-\tau_R)\delta + \tau_p\right] < 0. \qquad (15.69)$$

The general solution to (15.68) is

$$p_t = \left(p_{t_0} - (1-\tau_R)\int_{t_0}^t R(\tilde{H}_s)e^{a(s-t_0)}ds\right)e^{-a(t-t_0)}.$$

Multiplying through by $e^{a(t-t_0)}$ gives

$$p_t e^{a(t-t_0)} = p_{t_0} - (1-\tau_R)\int_{t_0}^{t} R(\tilde{H}_s)e^{a(s-t_0)}ds.$$

Rearranging and letting $t \to \infty$, we get

$$p_{t_0} = (1-\tau_R)\int_{t_0}^{\infty} R(\tilde{H}_s)e^{a(s-t_0)}ds + \lim_{t\to\infty} p_t e^{a(t-t_0)}.$$

Inserting (15.69), replacing t by T and t_0 by t, and comparing with (15.67), we see that

$$p_t = \hat{p}_t + \lim_{T\to\infty} p_T e^{-[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p](T-t)}. \tag{15.70}$$

The first term on the right-hand side is the fundamental value of the house at time t. The second term on the right-hand side thus amounts to a bubble, driven by self-fulfilling expectations. In the absence of the bubble, the market price, p_t, coincides with the fundamental value.

Are rational price bubbles possible? We see from (15.70) that a positive rational bubble being present requires that

$$\lim_{T\to\infty} p_T e^{-[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p](T-t)} > 0.$$

In turn, this requires that the house price is explosive in the sense of ultimately growing at a rate not less than $(1-\tau_r)r + (1-\tau_R)\delta + \tau_p$. Our candidate for a bubbly path ultimately moving North-East in Fig. 15.9 in fact has this property. Indeed, by (15.64), for such a path we have

$$\dot{p}_t/p_t = [(1-\tau_r)r + (1-\tau_R)\delta + \tau_p] - (1-\tau_R)R(\tilde{H}_t)/p_t \to (1-\tau_r)r+(1-\tau_R)\delta+\tau_p$$

for $t \to \infty$, since $p_t \to \infty$ and $R'(\tilde{H}_t) < 0$. But such an explosive price path can hardly be realized under rational expectations, as explained in the text of Section 15.2.4.

15.5 Exercises

(15.64)