Chapter 14

Fixed capital investment and Tobin's q

The models considered so far (the OLG models as well as the representative agent models) have ignored capital adjustment costs. In the closed-economy version of the models aggregate investment is merely a reflection of aggregate saving and appears in a "passive" way as just the residual of national income after households have chosen their consumption. We can describe what is going on by telling a story in which firms just rent capital goods owned by the households and households save by purchasing additional capital goods. In these models only households solve intertemporal decision problems. Firms merely demand labor and capital services with a view to maximizing current profits. This may be a legitimate abstraction in some contexts within long-run analysis. In short- and medium-run analysis, however, the dynamics of fixed capital investment is important. So a more realistic approach is desirable.

In the real world the capital goods used by a production firm are usually owned by the firm itself rather than rented for single periods on rental markets. One reason for this is that capital goods are often firm-specific, designed or at least adapted to the firm in which they are an integrated part. The capital goods are therefore generally worth more to the user than to others.

Tobin's *q-theory of investment* (after the American Nobel laureate James Tobin, 1918-2002) is an attempt to model these features. In this theory,

(a) *firms* make the *investment decisions* and *install* the purchased capital goods in their own businesses with the aim of maximizing discounted expected earnings in the future;

(b) there are certain *adjustment costs* associated with this investment: before acquiring new capital goods there are planning and design costs, and along with the implementation of the investment decisions there are costs of installation of the new equipment, costs of reorganizing the plant, costs of retraining workers to operate the new machines etc.;

(c) the adjustment costs are *strictly convex* in the sense that marginal adjustment costs are increasing in the level of investment − think of adding a side wing to a factory in a quarter of a year rather than a year.

The presence of capital adjustment costs fits well with the general notion that in the short run, the capital costs of firms are *fixed costs*. We observe that firms sometimes make losses. If all production costs were variable costs, this would not happen. The assumption of strict convexity of the capital adjustment costs captures the intuitive perception that to increase the capital stock by a given amount is more costly when doing it fast rather than slowly. We avoid the unrealistic picture of firms' capital as a production factor that can move instantaneously across firms and industries.

When faced with strictly convex installation costs, the optimizing firm has to take the *future* into account. So firms' forward-looking *expectations* become important. To smooth out the installation costs, the firm will adjust its capital stock only *gradually* when new information arises. From an analytical point of view, we thereby avoid the counter-factual implication in earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world financial market changes. Moreover, sluggishness in investment is what the data indicate. Some empirical studies conclude that only a third of the difference between the current and the "desired" capital stock tends to be covered within a year (Clark 1979).

The strictly convex adjustment costs assign investment decisions an *active* role in a macroeconomic model. There will be both a well-defined saving decision and a well-defined investment decision, separate from each other. Households decide the saving, firms the physical capital investment; households accumulate financial assets, firms accumulate physical capital. As a result, in a closed economy the current and expected future interest rates have to adjust for aggregate demand for goods (consumption plus investment) to match aggregate supply of goods. The role of interest rate changes is no longer to clear a rental market for capital goods.

Under certain conditions, to be described in Section 14.2, the theory leads to a remarkably simple operational macroeconomic investment function, in which the key variable explaining aggregate investment is the valuation of the firms by the stock market relative to the replacement value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

14.1 Convex capital installation costs

Let the technology of a single firm be given by

$$\tilde{Y} = F(K, L), \tag{14.1}$$

where $\tilde{Y}$, K, and L are "potential output" (to be explained), capital input, and labor input per time unit, respectively, while F is a (jointly) concave neoclassical production function. So we allow decreasing as well as constant returns to scale (or a combination of locally CRS and locally DRS), whereas increasing returns to scale is ruled out. Until further notice technological change is ignored for simplicity.

Time is continuous. The dating of the variables will not be explicit unless needed for clarity. The increase per time unit in the firm's capital stock is given by

$$\dot{K} = I - \delta K, \qquad \delta > 0, \qquad K_0 > 0, \tag{14.2}$$

where I is gross fixed capital investment per time unit and δ is the rate of wearing down of capital (physical capital depreciation). (To avoid having to concider specialties arising when $\delta = 0$, we rule this rare special case out.)

To fix the terminology, from now on the different adjustment costs associated with investment will as a rule be subsumed under the term *capital installation costs*. Let J denote these costs (measured in units of output) per time unit. The installation costs imply that a part of the potential output, $\tilde{Y}$, is "used up" in transforming investment goods into installed capital (possibly simply forgone due to interruptions of production during the process of installation). Only $\tilde{Y} - J$ is output available for sale.

Assuming the price of investment goods is one (the same as that of output goods), then total investment outlay per time unit are $I + J$, i.e., the direct purchase price, $1 \cdot I$, plus the indirect cost, J, associated with installation. The q-theory of investment assumes that the installation cost is a strictly convex function of gross investment and a non-increasing function of the current capital stock. Thus,

$$J = G(I, K),$$

where the installation cost function, G, is a $\mathbb{C}^2$ function satisfying

$$G(0, K) = 0, \; G_I(0, K) = 0, \; G_{II}(I, K) > 0, \text{ and } G_K(I, K) \leq 0 \tag{14.3}$$

for all pairs (I, K) with $I \gtreqless 0$ and $K \geq 0$, with the exception of pairs where $I < 0$ and $K = 0$. Negative gross investment (sell off of capital equipment) is possible,

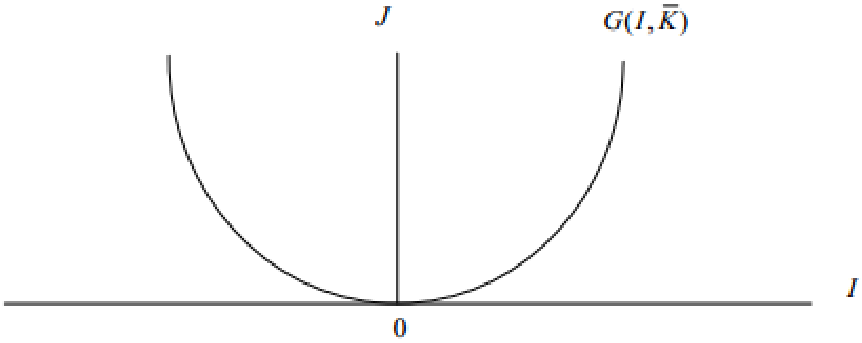

Figure 14.1: Installation costs as a function of gross investment when $K = \bar{K}$.

but of course only when $K > 0$. The required dismantling and reorganization involves adjustment costs, the more so the larger is $|I|$. Thus, $G_I < 0$ for $I < 0$ whereas $G_{II}(I, K) > 0$ for *any* I.

At the cost of some minor and uninteresting loss of generality, we add the assumption that G is a (jointly) convex function of (I, K). This means that, in addition to (14.3),

$$G_{KK} \geq 0 \text{ and } G_{II}G_{KK} - (G_{IK})^2 \geq 0 \tag{14.4}$$

for all (I, K).[1] Examples of G functions satisfying (14.3) and (14.4) are $G(I, K) = (\frac{1}{2})\beta I^2/K$ as well as the simpler $G(I, K) = (\frac{1}{2})\beta I^2$, where in both cases $\beta > 0$. Although for instance $G(I, K) = I^2/K$ does not seem to fit the condition $G(0, 0) = 0$, we *define* in this case $G(0, 0)$ as $\lim_{K\to 0^+}(0/K) = 0$. The latter equality holds in view of L'Hopital's rule for "0/0". Nevertheless, in the following we will concentrate on investment in already established firms ($K > 0$). This is to avoid complicating the analysis by discontinuities associated with start-up of firms.[2]

For fixed $K = \bar{K} > 0$, the properties of G are illustrated in Fig. 14.1. The important property is that $G_{II} > 0$ (strict convexity in I), implying that the marginal installation cost is increasing in the level of investment. If the firm wants to accomplish a given installation project in only half the time, then the installation costs are more than doubled (the risk of mistakes is larger, the problems with reorganizing work routines are larger etc.).

The strictly convex graph in Fig. 14.1 illustrates the essence of the matter. Assume the current capital stock in the firm is $\bar{K}$ and that the firm wants to

[1] At the end of Section 14.1.2 a certain technical regularity assumption on G is added for convenience.

[2] For instance, $G(I, K) = I^2/K$ will be *discontinuous* at $(0, 0)$, since for $I > 0$, $\lim_{K\to 0^+}(I^2/K) = \infty$ so that $\lim_{I\to 0^+}(\lim_{K\to 0^+}(I^2/K)) = \infty$, while, by definition, $G(0, 0) = 0$.

increase it by a given amount $\overline{\Delta K}$. If the firm chooses the investment level $\bar{I} > 0$ per time unit in the time interval $[t, t+\Delta t)$, then, in view of (14.2), $\Delta K \approx (\bar{I} - \delta\bar{K})\Delta t$. So it takes $\Delta t \approx \overline{\Delta K}/(\bar{I} - \delta\bar{K})$ units of time to accomplish the desired increase $\overline{\Delta K}$, and the installation costs are approximately $G(\bar{I}, \bar{K})\Delta t$. If, however, the firm slows down the adjustment and invests only half of $\bar{I}$ per time unit, then it takes approximately twice as long time to accomplish $\overline{\Delta K}$, but total costs are now approximately $G(\frac{1}{2}\bar{I}, \bar{K})2\Delta t$ (assuming, for simplicity, that $G_K(I, K) = 0$, and ignoring discounting). As illustrated in Fig. 14.1, the last-mentioned cost is smaller than the first-mentioned. This is due to the strict convexity of installation costs. *Haste is waste.*

On the other hand, there are limits to how slow the adjustment to the desired capital stock should be. Slower adjustment means postponement of the potential benefits of a higher capital stock. So the firm faces a trade-off between fast adjustment to the desired capital stock and low installation costs.

In addition to the strict convexity of G with respect to I, (14.3) imposes the condition $G_K(I, K) \leq 0$. A given amount of investment per time unit may require more reorganization in a small plant than in a large plant (measured by size of K). Owing to indivisibilities, when installing a new machine, a small firm has to stop production altogether, whereas a large firm can to some extent continue its production by shifting some workers to another production line. A further argument, but less accurate, is that the more a firm has invested historically, the more experienced it is now concerning how to avoid large installation costs. So, for a given I today, the associated installation costs are lower, given a larger accumulated K.

14.1.1 The decision problem of the firm

In the absence of tax distortions, asymmetric information, and problems with enforceability of financial contracts, the Modigliani-Miller theorem (Modigliani and Miller, 1958) entails that the financial structure of the firm is both indeterminate and irrelevant for production decisions (see Appendix A). Although the conditions required for validity of this theorem are quite idealized, the q-theory of investment accepts them as a starting point, allowing the analyst to concentrate on the production aspects in a first approach.

We assume that the output good as well as the investment good has the price 1 throughout. Let the operating cash flow (the net payment stream to the firm before interest payments on debt, if any) at time t be denoted R_t (for net "receipts"). Then

$$R_t \equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \qquad (14.5)$$

where the wage rate at time t is denoted w_t. As mentioned, the installation cost $G(I_t, K_t)$ implies that a part of production, $F(K_t, L_t)$, is used up in transforming investment goods into installed capital. Only the difference $F(K_t, L_t) - G(I_t, K_t)$ is available for sale. This sacrifice of a part of potential output arises from the need to devote some of the firms' resources to the installation of new machines.

Throughout this chapter we assume the firm is a price taker in the labor market. At the start it is also a price taker in the output market. We ignore uncertainty.The interest rate is r_t, which we assume is positive at least in the long run. Given our normalization of prices, w_t and r_t are to be interpreted as a real wage and real interest rate, respectively. The decision problem, as seen from time 0, is the following: given the expected evolution of market prices, $(w_t, r_t)_{t=0}^{\infty}$, choose a plan $(L_t, I_t)_{t=0}^{\infty}$ so as to maximize the firm's *market value*, i.e., the present (discounted) value of the future stream of expected cash flows:

$$\max_{(L_t,I_t)_{t=0}^{\infty}} V_0 = \int_0^{\infty} R_t e^{-\int_0^t r_\tau d\tau} dt \quad \text{s.t. (14.5) and} \tag{14.6}$$

$$L_t \geq 0, I_t \text{ “free” (i.e., no restriction on } I_t), \tag{14.7}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{14.8}$$

$$K_t \geq 0 \text{ for all } t. \tag{14.9}$$

There is no specific terminal constraint but we have posited the feasibility condition (14.9) saying that the firm can never have a negative capital stock.[3]

In the previous chapters the firm was described as solving a series of static profit maximization problems. Such a description is no longer valid, however, when there is dependence across time, as is the case here. When installation costs are present, current decisions depend on the expected future circumstances. What the firm can control at each moment is the rate of investment, not the stock of capital. The firm makes a plan for the whole future so as to maximize the value of the firm, which is what matters for the owners. This is the general neoclassical hypothesis about firms' investment behavior. We use the terms *value maximization* and *intertemporal profit maximization* synonymously.[4]

[3]It is assumed that w_t and r_t are piecewise continuous functions. At points of discontinuity (if any) in investment, we consider investment to be a *right-continuous* function of time. That is, $I_{t_0} = \lim_{t \to t_0^+} I_t$. Likewise, at such points of discontinuity, by the "time derivative" of the corresponding state variable, K, we mean the *right-hand* time derivative, i.e., $\dot{K}_{t_0} = \lim_{t \to t_0^+} (K_t - K_{t_0})/(t - t_0)$. Mathematically, these conventions are inconsequential, but they help the intuition.

[4]When strictly convex installation costs, or other dependencies across time, are *absent*, then, as shown in Appendix A, value maximization is equivalent to solving a sequence of isolated *static* profit maximization problems, and we are back in the previous chapters' description.

To solve the problem (14.6) − (14.9), where R_t is given by (14.5), we apply the Maximum Principle. The problem has two control variables, L and I, and one state variable, K. We set up the current-value Hamiltonian:

$$H(K, L, I, q, t) \equiv F(K, L) - wL - I - G(I, K) + q(I - \delta K), \tag{14.10}$$

where q (to be interpreted economically below) is the adjoint variable associated with the dynamic constraint (14.8). For each $t \geq 0$ we maximize H with respect to the control variables. Thus, $\partial H/\partial L = F_L(K, L) - w = 0$, i.e.,

$$F_L(K, L) = w, \tag{14.11}$$

and $\partial H/\partial I = -1 - G_I(I, K) + q = 0$, i.e.,

$$1 + G_I(I, K) = q. \tag{14.12}$$

Next, we partially differentiate H with respect to the state variable and set the result equal to $rq - \dot{q}$, where r is the discount rate in (14.6):

$$\frac{\partial H}{\partial K} = F_K(K, L) - G_K(I, K) - q\delta = rq - \dot{q}. \tag{14.13}$$

Then, the Maximum Principle says that for an interior optimal path $(K_t, L_t, I_t)_{t=0}^{\infty}$, there exists an adjoint variable q, which is a continuous function of t, written q_t, such that for all $t \geq 0$ the conditions (14.11), (14.12), and (14.13) hold along the path. Moreover, it can be shown that the path will for all $t \geq 0$ have $q_t \geq 0$ and satisfy the "standard" infinite horizon transversality condition

$$\lim_{t \to \infty} K_t q_t e^{-\int_0^t r_\tau d\tau} = 0. \tag{14.14}$$

The optimality condition (14.11) is the usual employment condition equalizing the marginal productivity of labor to the real wage. In the present context with strictly convex capital installation costs, this condition attains a distinct role as labor will in the short run be the only variable input. Indeed, the firm's installed capital is in the short run a fixed production factor due to the strictly convex capital installation costs. So, effectively there are diminishing returns (equivalent to rising marginal costs) in the short run even though the production function might have CRS.

The left-hand side of (14.12) gives approximately the extra cost associated with increasing the investment level by one unit per time unit. This extra investment cost will be the sum of the purchase price of the investment good, here 1, and the rise in total installation costs per time unit it causes. The left-hand side is thus the marginal procurement cost, MC, of capital in the firm. Since (14.12) is a necessary condition for optimality, the right-hand side of (14.12) must then be the marginal benefit, MB, of installed capital along the optimal path. So q_t represents the value to the optimizing firm of having one more unit of installed capital at time t. To put it differently: the adjoint variable q_t can be seen as the *shadow price* (measured in current output units) of capital along the optimal path.[5] Thereby q_t can also be seen as the total *cost saving* associated with reducing the investment by one unit. In this situation the firm recovers q_t by saving on installation costs as well as the one-time cost to buy a new machine.

Continuing along this line of thought, by reordering in (14.13), we get the "no-arbitrage" condition

$$F_K(K, L) - G_K(I, K) - \delta q + \dot{q} = rq, \tag{14.15}$$

saying that along the optimal path, whenever $q_t > 0$, the rate of return on the marginal unit of installed capital must equal the interest rate. Rearranging, the condition says that the firm at every t has acquired capital up to the point where the "total marginal productivity of capital", $F_K - G_K$, equals the imputed marginal "operating cost of capital", $r_t q_t + (\delta q_t - \dot{q}_t)$. The first term in this cost represents imputed interest cost and the second term the imputed economic depreciation. The "total marginal productivity of capital" appears as $F_K - G_K$, thereby taking into account the reduction, $-G_K$, of installation costs brought about by the marginal unit of installed capital. From a "technical" point of view, the importance of (14.13) is that it delivers a differential equation for the shadow price q to supplement the differential equation for K given by the constraint (14.8).

The transversality condition (14.14) says that along an optimal path the present value of the state variable "left over" at infinity must be zero. Such a terminal condition is necessary for optimality in many economic problems with infinite horizon, including the present one where the state variable is the stock of capital.[6] Valued capital "left over" at eternity is like "money left on the table". So violation of (14.14) means "overinvesting": in the long run investment and installation costs exceeds the benefits of the extra installed capital.

It can also be shown that, irrespective the time path of the capital stock, optimality requires that the present value of the shadow price itself, when discounted by $r + \delta$, is asymptotically zero, i.e.,

$$\lim_{t\to\infty} q_t e^{-\int_0^t (r_\tau + \delta) d\tau} = 0. \tag{14.16}$$

[5] Recall that a *shadow price,* measured in some unit of account, of a good, from the point of view of the buyer, is the maximum number of units of account that he or she is willing to offer for one extra unit of the good.

[6] A proof is given in Appendix B.

If along an optimal path, K_t is declining in the long run, the condition (14.16) gives additional information relatve to (14.14).

In connection with (14.12) we claimed that q_t can be interpreted as the shadow price (measured in current output units) of capital along the optimal path. A confirmation of this interpretation is obtained by solving the differential equation (14.13). Indeed, multiplying by $e^{-\int_0^t (r_\tau+\delta)d\tau}$ on both sides of (14.13), we get by integration and application of (14.16),[7]

$$q_t = \int_t^\infty \left[F_K(K_s, L_s) - G_K(I_s, K_s)\right] e^{-\int_t^s (r_\tau+\delta)d\tau} ds > 0. \tag{14.17}$$

The right-hand side here is the present value, as seen from time t, of the expected future increases of the firm's cash-flow that would result if one extra unit of capital were installed at time t. Indeed, $F_K(K_s, L_s)$ is the direct contribution to output of one extra unit of capital, while $-G_K(I_s, K_s) \geq 0$ represents the reduction of installation costs in the next instant brought about by the marginal unit of installed capital. Note that the marginal future increases of cash-flow in (14.17) are discounted at a rate equal to the interest rate *plus* the capital depreciation rate. The reason is that from one extra unit of capital at time t there are only $e^{-\delta(s-t)}$ units left at time s. The inequality in (14.17) is ensured because we consider an *interior* optimal path and $F_K > 0$, while $G_K \leq 0$.

To concretize our interpretation of q_t, let us make a thought experiment. Assume that a extra units of installed capital at time t drops down from the sky. At time $s > t$ there are $a \cdot e^{-\delta(s-t)}$ units of these still in operation so that the stock of installed capital is

$$K_s' = K_s + a \cdot e^{-\delta(s-t)}, \tag{14.18}$$

where K_s denotes the stock of installed capital as it would have been without this "injection". In (14.5), imagine t is replaced by s and consider the optimizing firm's cash-flow R_s as a function of (K_s, L_s, I_s, s, t, a). Taking the partial derivative of R_s with respect to a at the point $(K_s, L_s, I_s, s, t, 0)$, gives

$$\frac{\partial R_s}{\partial a}_{|a=0} = \left[F_K(K_s, L_s) - G_K(I_s, K_s)\right] e^{-\delta(s-t)}. \tag{14.19}$$

Next we consider the value of the optimizing firm at time t as a function of installed capital, K_t, and t itself. This function is called the *(optimal) value function.* We write it $V^*(K_t, t)$. Intuitively, we have

$$\begin{aligned} \frac{\partial V^*(K_t, t)}{\partial K_t} &= \int_t^\infty \left(\frac{\partial R_s}{\partial a}_{|a=0}\right) e^{-\int_t^s r_\tau d\tau} ds \\ &= \int_t^\infty [F_K(K_s, L_s) - G_K(I_s, K_s)] e^{-\int_t^s (r_\tau+\delta)ds} ds = q_t, \end{aligned} \tag{14.20}$$

[7] For details, see Appendix B.

when the firm moves along the optimal path.[8] The second equality sign comes from (14.19) and the third is implied by (14.17). We see that the value of the adjoint variable, q, at time t equals the contribution to the firm's maximized value of a fictional marginal "injection" of installed capital at time t. In brief: the shadow price q_t equals the present value of expected future "marginal operating profits".

These considerations provide economic intuition to the Maximum Principle saying that the control variables at any point in time should be chosen so that the Hamiltonian function is maximized. Thereby one maximizes the properly weighted sum of the immediate contribution to the criterion function and the indirect contribution, which is the benefit (as measured approximately by $q_t \Delta K_t$) of having a higher capital stock in the future.

As we know, the Maximum Principle gives only necessary conditions for an optimal path, not sufficient conditions. We use the principle as a tool for finding candidates for a solution. Having found in this way a candidate, one way to proceed is to check whether *Mangasarian's sufficient conditions* are satisfied. Given the transversality condition (14.14) and the non-negativity of the state variable, K, the only additional condition to check is whether the Hamiltonian function in (14.10) is for every t (jointly) concave in the endogenous variables that enter (here K, L, and I). This is indeed satified since the Hamiltonian function is a sum of concave functions (note that $-G(I, K)$ is concave in (I, K) since we assumed $G(I, K)$ itself to be convex in (I, K)). It follows that the first-order conditions together with the transversality condition are not only necessary, but also sufficient for an optimal solution.

14.1.2 The implied investment function

From the first-order condition (14.12) we can derive an investment function. Rewriting (14.12), we have that an optimal path satisfies

$$G_I(I_t, K_t) = q_t - 1. \tag{14.21}$$

Combining this with the assumption (14.3) on the installation cost function, we see that

$$I_t \gtreqless 0 \text{ for } q_t \gtreqless 1, \text{ respectively,} \tag{14.22}$$

[8]The qualification "intuitively" is motivated by the fact that, in general, optimal control theory does not guarantee differentiability of the value function in every point. In particular, an equality like the first one in the upper line of (14.20) need not hold because generally, an arbitrarily small change in the initial value of the state variable *may* change the whole optimal path qualitatively. But this kind of difficulty does not arise in the present problem. This is shown in Appendix D for the case where the functions F and G are homogeneous of degree one. For the general case, see Weitzman, 2003, Ch. 3.

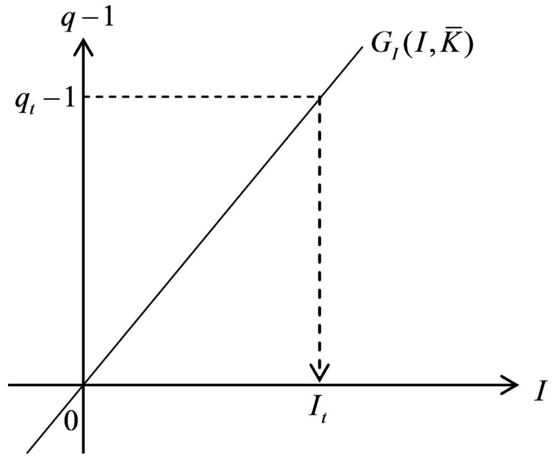

Figure 14.2: Marginal installation costs as a function of the gross investment level, I, for a given amount, $\bar{K}$, of installed capital. The optimal gross investment for $q = q_t > 1$ and for $q = q_t' \in (0,1)$ are indicated.

cf. Fig. 14.2.[9] By the implicit function theorem, in view of $G_{II} \neq 0$, (14.21) defines investment, I_t, as an implicit function of the shadow price, q_t, and the state variable, K_t,

$$I_t = \mathcal{M}(q_t, K_t), \tag{14.23}$$

with partial derivatives

$$\frac{\partial I_t}{\partial q_t} = \frac{1}{G_{II}(\mathcal{M}(q_t, K_t), K_t)} > 0, \qquad \text{and} \qquad \frac{\partial I_t}{\partial K_t} = -\frac{G_{IK}(\mathcal{M}(q_t, K_t), K_t)}{G_{II}(\mathcal{M}(q_t, K_t), K_t)}, \tag{14.24}$$

The latter cannot be signed without further specification. In view of (14.22), we have $\mathrm{M}(1, K_t) = 0$.[10]

It follows that optimal investment is an increasing function of the shadow price of installed capital. In view of (14.22), $\mathcal{M}(1, K) = 0$. Qualitatively, the investment rule is: invest now, if and only if the value to the firm of the marginal unit of installed capital is larger than the price of the capital good (which is 1, excluding installation costs). Quantitatively, the rule says that, because of

[9] From the assumptions made in (14.3), we only know that the graph of $G_I(I, \bar{K})$ is an upward-sloping curve going through the origin. Fig. 14.2 shows the special case where this curve happens to be linear.

[10] A method for calculating the formulas without consulting the implicit function theorem is to apply *implicit differentiation* in (14.21). That is, *first*, replace the dependent variable, I_t, in (14.21) by the implicit function $\mathcal{M}(q_t, K_t)$. *Second*, calculate the partial derivative with respect to q_t on both sides of (14.21), using the chain rule. *Third*, by rearranging solve for $\partial I_t / \partial q_t$. *Fourth*, calculate in a similar way the partial derivative with respect to K_t on both sides of (14.21) and solve for $\partial I_t / \partial K_t$.

the convex installation costs, invest only up to the point where the marginal installation cost, $G_I(I_t, K_t)$, equals $q_t - 1$, cf. (14.21).

Unfortunately, the investment equation (14.23) is not a recursive formula from which the firm – or the economic researcher – can directly infer the optimal investment level. At time t the current capital stock K_t is given and observable, but the firm's q_t is not. So (14.23) is a relationship between two unknowns, I_t and q_t. As we shall see, to pin down the firm's optimal q_t is a complicated dynamic problem – for the firm as well as for the researcher. An intertemporal mutual dependency between q and K is involved. While (14.23) tells us that optimal current investment depends on q_t, the equation (14.17) tells us that q_t depends on the expected future "marginal operating profits", and these will depend on investment today.

Still, when through a dynamic analysis, taking the expected future environment into account, the firm's "true", or optimal, q_t, has been found, it is remarkable how much information it contains. All the information about the production function, input prices, and interest rates now and in the future, which is relevant to the investment decision, is summarized in one number, q_t. Knowing this, together with the installation cost function G, the firm knows its optimal level of investment (since, when G is known, so is also the function $\mathcal{M}$, which is the inverse of G_I with respect to q, given K).

It remains to clarify the domain of the implicit "investment function" $\mathcal{M}$. At the cost of only uninteresting loss of generality, we add a regularity assumption concerning G to ensure that the domain of $\mathcal{M}$ includes all relevant pairs (q, K):

$$\text{for all } K > 0, \lim_{I \to \infty} G_I(I, K) = \infty \text{ and } \lim_{I \to -\infty} G_I(I, K) < -1. \tag{A1}$$

The absence of an upper bound for $G_I(I, K)$ ensures that for any value of $q \geq 1$, there is a unique *nonnegative* investment level, I, satisfying (14.21). The second part of (A1) ensures that for any $q \in (0, 1)$, there exists, for a given $K > 0$, a unique *negative* investment level (disinvestment) satisfying (14.21). In this case the worth to the firm of the marginal unit of installed capital is positive but less than 1. While the marginal unit can be sold at the selling price 1, this requires to first defray the dismantling costs, $G(I, K)$, associated with the chosen level of negative investment, given K. This level, I_1 say, will satisfy the optimality condition $1 + G_I(I_1, K) = 1 - |G_I(I_1, K)| = q < 1$.

Technical Remark. What about $q \leq 0$? At $q = 0$, the firm would be indifferent between dismantling (choosing $I_1 < 0$ such that $|G_I(I_1, K)| = 1$) and $I = 0$, and so $\mathcal{M}$ is not defined. A negative q is impossible, because free disposal rules out a negative shadow price of an asset. □

As an example consider the following special case.

The capital adjustment principle Suppose the installation costs are $J = \tilde{G}(I) = (\beta/2)I^2$, $\beta > 0$, and let F have CRS and satisfy the Inada conditions. Let the interest rate be a constant $r > 0$ and the wage rate a constant $w > 0$.

We than have $Y = F(K, L) = LF(k, 1) \equiv Lf(k)$, where $k \equiv K/L$. By first-order condition (14.11), $F_L(K, L) = f(k) - kf'(k) \equiv \psi(k) = w$. Since $\psi'(k) = -kf''(k) > 0$, this allows writing $k = \psi^{-1}(w) \equiv k(w)$ with

$$k'(w) = \frac{1}{\psi'(k(w))} = -\frac{1}{k(w)f''(k(w))} > 0, \tag{14.25}$$

which is to be used in a moment. At every t, K_t is given, and the firm will chose L_t such that $K_t/L_t = k$, where k is a constant given by $k = k(w)$. Note also that $F_K(K_t, L_t) = f'(k(w))$ for all t.

The first-order condition (14.12) reads $\tilde{G}'(I_t) = \beta I_t = q_t - 1$, implying the investment function $I_t = (q_t - 1)/\beta \equiv \mathcal{M}(q_t)$. The differential equation for K_t thus is

$$\dot{K}_t = (q_t - 1)/\beta - \delta K_t, \qquad K_0 > 0 \text{ given.} \tag{*}$$

So

$$\dot{K}_t \gtreqqless 0 \text{ for } q_t \gtreqqless 1 + \beta\delta K_t, \quad \text{respectively.}$$

The differential equation for q is obtained by rearranging the first-order condition (14.13) to get

$$\dot{q}_t = (r + \delta)q_t - F_K(K_t, L_t) = (r + \delta)q_t - f'(k(w)), \tag{**}$$

implying

$$\dot{q}_t \gtreqqless 0 \text{ for } q_t \gtreqqless \frac{f'(k(w))}{r + \delta} \equiv q^*(w), \quad \text{respectively,}$$

where we have suppressed the dependency of q^* on the parameters r and δ. Finally, the necessary transversality condition is

$$\lim_{t\to\infty} K_t q_t e^{-rt}.$$

To determine the optimal investment path, we (as well as the firm) need the "true" shadow price, i.e., the optimal initial value of q. To find this value, we construct the phase diagram shown in Fig. 14.3. We assume that $f'(k(w))/(r+\delta) > 1$, because existence of a positive steady-state value for K, K^*, requires that $q^*(w) \equiv f'(k(w))/(r + \delta) = 1 + \beta\delta K^* > 1$.

The direction of movement in the different regions of the phase diagram are determined by (*) and (**) and is indicated by arrows in Fig. 14.3. The arrows taken together show that the steady state is a saddle point. In this example the saddle path is horizontal and coincides with the $\dot{q} = 0$ locus. As the capital stock is pre-determined, initially the firm must be situated at some point on the vertical line $K = K_0$ in the figure. A reasonable supposition is that the optimal initial value of q is the ordinate to the point of intersection between this line and the saddle path, i.e., the point A. This ordinate is q^*. With this initial q, the firm will follow the saddle path over time and converge to the steady state, E. In the steady state, the transversality condition is clearly satisfied since $r > 0$. The same holds for any path converging to the steady state, whether $K_0 < K^*$, as in the figure, or $K_0 > K^*$. Since the Hamiltonian is concave in (L, I, K), by Mangasarian's sufficiency theorem follows that the considered path *is* an optimal path. Could there be *other* solutions? No! Paths starting *above* the saddle path will have $\dot{q}_t/q_t = r + \delta - f'(k(w))/q_t \to r + \delta$ for $t \to \infty$. At least ultimately, also K will be growing, and then $K \cdot q$ will ultimately grow at a rate higher than r. The transversality condition will thus be violated. Paths starting *below* the saddle path will for all t have q below $q^*(w)$ and thus not fully exploit the potential benefits of capital.[11]

The unique optimal investment plan is thus

$$I_t = \frac{1}{\beta}(q^*(w) - 1) = \frac{1}{\beta}\left(\frac{f'(k(w))}{r+\delta} - 1\right) \equiv \bar{I} \qquad \text{for all } t \geq 0,$$

thereby maintaining gross investment constant. This is the *investment function* in the sense of a relationship giving optimal investment as a function of exogenous variables and parameters. The resulting capital dynamics is $\dot{K}_t = \bar{I} - \delta K_t$, and this linear differential equation has the solution

$$K_t = (K_0 - K^*)e^{-\delta t} + K^*, \qquad K^* = \frac{\bar{I}}{\delta} = \frac{1}{\delta\beta}\left(\frac{f'(k(w))}{r+\delta} - 1\right). \qquad (***)$$

What explains the constancy of optimal gross investment? We know from (14.17) that the shadow price of installed capital equals the present value of expected future marginal profits along the optimal plan. Thus, in this example,

[11] This argument can be unfolded in detail by applying the general formula for the value function $V^*(K_t, t)$ given in (14.67) of Appendix D, letting $t = 0$. Anyway, uniqueness of the solution to dynamic optimization problems in economics is common. In particular, uniqueness is to be expected when, as here, the Hamiltonian is at every t strictly concave in the control vector, here (L, I) (and uniqueness would be ensured if the Hamiltonian is strictly concave in (L, I, K), but that is not the case here).

It would be incorrect, however, to rule out optimality of a path starting with $q_0 < q^*$ by arguing that such a path, in view of (**), will violate the transversality condition in that q ultimately becomes *negative* and declines faster and faster. The truth is that the shadow price q can *never* become negative, cf. Technical Remark above. The differentials equations (*) and (**) are thus only valid as long as $q_t > 0$. After q has reached zero, q remains at zero while K gradually declines towards zero.

where $G_K \equiv 0$,

$$q_t = \int_t^\infty f'(k(w))e^{-(r+\delta)(s-t)}ds = \frac{f'(k(w))}{r+\delta},$$

which is independent of t and is identical to our $q^*(w)$. Then the first-order condition (14.12) reduces to $1 + \beta I_t = q^*(w)$, and so optimal gross investment, I_t, must be constant. The *intuition* behind the constancy of gross investment is that the strictly convex installation costs implies an incentive to *smooth out* the investment.

Optimal *net* investment is

$$I_t^n \equiv I_t - \delta K_t = \bar{I} - \delta K_t = \delta K^* - \delta K_t = \delta(K^* - K_t). \qquad (****)$$

This relationship has traditionally been called the *capital adjustment principle* because it can be interpreted as describing the gradual adjustment of actual to "desired capital", K^*. The principle says that optimal net investment is proportional to the distance between desired and actual capital. Net investment is positive (negative) as long as the actual capital stock is below (above) the desired capital stock. This is so whatever the size of r. There is thus no stable relationship between *net* investment and r.

How large is the *speed* of adjustment ($\equiv$ the rate of decline of the distance to the steady state)? In view of $d(K_t - K^*)/dt = \dot{K}_t = I_t^n = \delta(K^* - K_t)$, by (****), the speed of adjustment is constant and equal to the rate of (physical) depreciation, δ. Why? Because, for the steady-state level of capital, K^*, to be consistent with the level of gross investment, $\bar{I}$, K^* must be such that depreciation per time unit, δK^*, equals $\bar{I}$.

The strictly convex installation costs thus provide a "micro foundation" of the capital adjustment principle. In the absence of these costs, the desired capital would be reached immediately by acquiring capital in a bulk. Mathematically, this would amount to an upward *jump* in K. As the capital formation technique is formulated in the decision problem of the firm, cf. (14.8), this is impossible. An interpretation is that it is infinitely costly.

In steady state, as well as during the approach towards steady state, in view of $q^*(w) > 1$, net marginal productivity of capital exceeds the interest rate:

$$F_K(K, L) = f'(k(w)) = (r+\delta)q^*(w) > r + \delta.$$

In spite of this, there is no incentive to increase K further. The reason is that the marginal cost of doing so exceeds the marginal benefit due to the installation costs. Note also that, by (***),

$$\begin{aligned}
\frac{\partial \bar{I}}{\partial w} &= \frac{\partial(\delta K^*)}{\partial w} = \delta\frac{\partial K^*}{\partial w} = \frac{f''(k(w))k'(w)}{\beta(r+\delta)} = \frac{-1}{\beta(r+\delta)k(w)} < 0, \quad \text{(from (14.25))} \\
\frac{\partial \bar{I}}{\partial r} &= \frac{\partial(\delta K^*)}{\partial r} = \delta\frac{\partial K^*}{\partial r} = -\frac{f'(k(w))}{\beta(r+\delta)^2} < 0,
\end{aligned}$$

Gross investment and the desired capital stock thus depend negatively on both cost variables, w and r. This also follows graphically by curve shifting in Fig. 14.3.

Considering the role of w, the positive *substitution effect* on K/L of a higher w is (under perfect competition) more than neutralized by a negative *level effect* (on the "desired capital stock") of the higher cost of production implied by the higher labor cost. The explanation is that the present value of expected future marginal profits is reduced by the higher cost of production.

Note that the capital adjustment principle builds on restrictive assumptions. The idea of a constant desired capital stock seems best fitted to a stationary economy. What should in particular be emphasized is that the analysis above is only *partial analysis.* The focus is on the behavior of a single firm in a *given* simplified economic "environment" – a constant arbitrary wage rate and a constant arbitrary interest rate. When *many* firms act in a similar environment, the aggregate result is usually that this "environment" is affected. So for instance the assumed constancy of the wage rate can no longer be maintained, and the dynamics of q is thus affected *qualitatively.* Taking these feedbacks – and feedbacks on feedbacks – into account implies a shift to *general equilibrium analysis* of firms' investment behavior. This is our focus in Section 14.3. □

(new Fig. 14.3 about here)

14.1.3 When G is homogeneous of degree one

While variation in the stock of capital is seldom quantitatively important in short-run analysis, it really matters in long-run analysis. We will now concentrate on the case where not only does K matter for the installation costs, but the installation cost function G is homogeneous of degree one with respect to I and K. For $K > 0$ we thus have

$$\begin{aligned}
J &= G(I,K) = G(\frac{I}{K},1)K \equiv g(\frac{I}{K})K, \qquad \text{or} \qquad (14.26) \\
\frac{J}{K} &= g(\frac{I}{K}),
\end{aligned}$$

where $g(\cdot)$ represents the *installation cost-capital ratio.*

LEMMA 1 Let the function G be homogeneous of degree one in addition to satisfying (14.3). Then the function $g(I/K) \equiv G(I/K, 1)$ has the following properties:
(i) $g(0) = 0$;
(ii) $g'(I/K) = G_I(I, K) \gtreqless 0$ for $I \gtreqless 0$, respectively;
(iii) $g''(I/K) = G_{II}(I, K)K > 0$ for $K > 0$;
(iv) $g(I/K) - g'(I/K)I/K = G_K(I, K) = G_K(I/K, 1) < 0$ for $I \neq 0$.

Proof. (i) $g(0) \equiv G(0, 1) = 0$. (ii) $G_I = Kg'/K = g'$. (iii) $G_{II} = g''/K$. (iv) $G_K = \partial(g(I/K)K)/\partial K = g(I/K) - g'(I/K)I/K$. Moreover, in view of $g'' > 0$ and $g(0) = 0$, we have $g(x) < g'(x)x$ for all $x \neq 0$. Finally, $G_K(I, K) = G_K(I/K, 1)$ follows from Euler's theorem, saying that when G is homogeneous of degree 1, then the partial derivatives of G are homogeneous of degree 0. □

The graph of $g(I/K)$ is qualitatively the same as that in Fig. 14.1 (imagine we have $\bar{K} = 1$ in that graph). The installation cost relative to the existing capital stock is now a strictly convex function of the investment-capital ratio, I/K. Note that in relation to our original characterization of the adjustment cost function G in (14.3), the only qualitative modification implied by the homogeneity of degree one is that the property $G_K \leq 0$ is sharpened to $G_K < 0$ except when $I = 0$.

A further important property of (14.26) is that the cash-flow function in (14.5) becomes homogeneous of degree one with respect to K, L, and I in the "normal" case where the production function has CRS. This has two implications. First, Hayashi's theorem applies (see below). Second, the q-theory can easily be incorporated into a model of economic growth.[12]

Does the hypothesis of homogeneity of degree one of the cash flow in K, L, and I make economic sense? According to a standard replication argument it does. Suppose a given firm has K units of installed capital and produces Y units of output with L units of labor. When at the same time the firm invests I units of account in new capital, it obtains the cash flow R after deducting the installation costs, $G(I, K)$. Then it makes sense to assume that the firm could do the same thing at another place, hereby doubling its cash-flow. (Of course, owing to the possibility of indivisibilities, this reasoning does not take us all the way to homogeneity of degree one. Moreover, the argument ignores that also land is a necessary input. As discussed in Chapter 2, in spite of mixed empirical evidence, the assumption of constant returns to scale with respect to capital and labor is in macroeconomics generally considered an acceptable approximation with regard to industrialized economies.)

[12] The relationship between the function g and other ways of formulating the theory in the literature is commented on in Appendix C.

In view of (i) of Lemma 1, the homogeneity-of-degree-one assumption for G allows us to write (14.21) as

$$g'(I/K) = q - 1. \tag{14.27}$$

This equation defines the investment-capital ratio, I/K , as an implicit function, m, of q :

$$\frac{I}{K} = m(q), \quad \text{where} \quad m(1) = 0 \quad \text{and} \quad m'(q) = \frac{1}{g''(m(q))} > 0, \tag{14.28}$$

by implicit differentiation in (14.27) and rearranging. We see that q encompasses all information that is of relevance to the decision about the investment-capital ratio.

EXAMPLE 1 Let $J = G(I, K) = (\beta/2)I^2/K$, where $\beta > 0$. Then G is homogeneous of degree one with respect to I and K and gives $J/K = (\beta/2)(I/K)^2 \equiv g(I/K)$, hence, $g'(I/K) = \beta I/K$. Then, by (14.27), $I/K = (q-1)/\beta \equiv m(q)$. And finally, by (iv) of Lemma 1, $G_K = -(q-1)^2/(2\beta)$ $(= -g(I/K))$. □

In this example $m(q)$ is linear, as illustrated in Fig. 14.3. The parameter β can be interpreted as the degree of sluggishness in the capital adjustment. The degree of sluggishness reflects the degree of convexity of installation costs.[13] Generally the graph of the investment function is positively sloped, but not necessarily linear. The interpretation of the stippled lines and q^* and n in Fig. 14.3 is as follows. Suppose the firm's employment grows at a constant rate $n \geq 0$. Then a constant capital-labor ratio, K/L, requires $\dot{K}/K = n$, hence $I/K - \delta = m(q) - \delta = n$. The investment-capital ratio, I/K, required to match not only depreciation at rate δ but also employment growth at rate n is thus $\delta + n$. The level of q required to *motivate* such an investment-capital ratio is denoted q^* in the figure.

By (iv) of Lemma 1, when G is homogeneous of degree 1, we have

$$G_K(I, K) = g(\frac{I}{K}) - g'(\frac{I}{K})\frac{I}{K} = g(m(q)) - (q-1)m(q) \equiv -\varphi(q), \tag{14.29}$$

where the second equality comes from (14.28) and (14.27). Recall that $-G_K(I_t, K_t)$, hence $\varphi(q)$, indicates how much *lower* the installation costs approximately are due to the marginal unit of installed capital.

LEMMA 2 Let the function G be homogeneous of degree one in addition to satisfying (14.3). Let the function φ be defined as in (14.29). Then:

[13] For a twice differentiable function, $f(x)$, with $f'(x) \neq 0$, we define the *degree of convexity* in the point x by $f''(x)/f'(x)$. So the degree of convexity of $g(I/K)$ is $g''/g' = (I/K)^{-1} = \beta(q-1)^{-1}$ and thereby we have $\beta = (q-1)g''/g'$. So, for given q, the degree of sluggishness is proportional to the degree of convexity of adjustment costs.

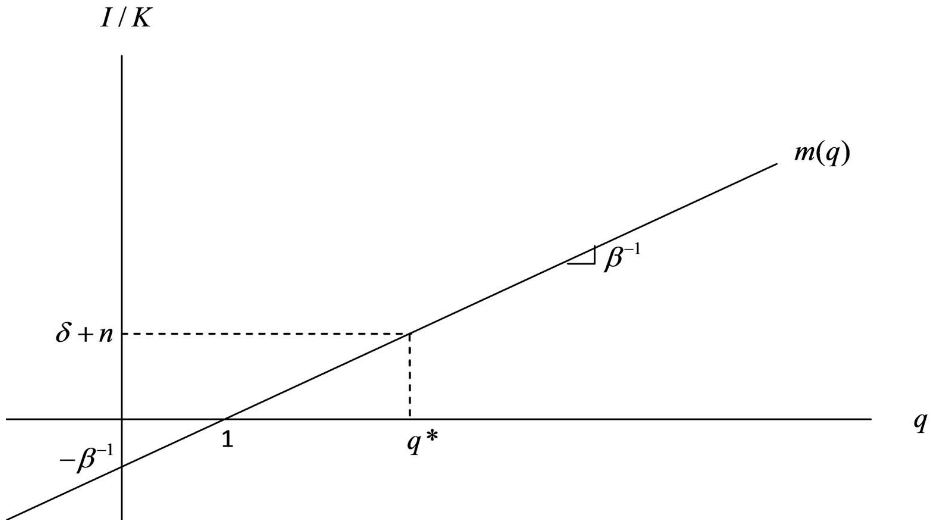

Figure 14.3: Optimal investment-capital ratio as a function of the shadow price of installed capital when $g(I/K) = \frac{1}{2}\beta(I/K)^2$.

(i) $\varphi(q) > 0$, when $q \neq 1$. In the special case $G(I,K) = (\beta/2)I^2/K$, $\beta > 0$, $\varphi(q) = (q-1)^2/(2\beta)$.
(ii) $\varphi'(q) = m(q) \gtreqless 0$ for $q \gtreqless 1$, respectively.

Proof. (i) That $\varphi(q) > 0$ when $q \neq 1$, follows from the definition in (14.29) combined with (iv) of Lemma 1 and (14.28); for the special case apply that $\varphi(q) \equiv -G_K(I,K)$. (ii) $\varphi'(q) = (q-1)m'(q) + m(q) - g'(m(q))m'(q) = m(q)$ since $g'(m(q)) = q - 1$ by (14.28) and (14.27). □

To see the implication for how the shadow price q changes over time along the optimal path, we first rearrange (14.13):

$$\dot{q}_t = (r_t + \delta)q_t - F_K(K_t, L_t) + G_K(I_t, K_t). \tag{14.30}$$

When G be homogeneous of degree one, we have from (14.29)

$$\begin{aligned} \dot{q}_t &= (r_t + \delta)q_t - F_K(K_t, L_t) + g(m(q_t)) - (q_t - 1)m(q_t) \\ &\equiv (r_t + \delta)q_t - F_K(K_t, L_t) - \varphi(q_t), \end{aligned} \tag{14.31}$$

This differential equation for the shadow price q_t is very useful in macroeconomic analysis, as we will soon see, cf. Fig. 14.4 below.

We now consider an example with technical progress in production.

A growing firm under perfect competition In case of technical change affecting installation costs, we should write the installation costs as $J_t = G(I_t, K_t, t)$.

We ignore this possible extension, and let the installation costs be as in Example 1 above. Thus $I_t/K_t = (q_t - 1)/\beta$, and

$$\dot{K}_t = \left(\frac{q_t - 1}{\beta} - \delta\right) K_t, \qquad K_0 > 0 \text{ given.} \tag{*}$$

The production technology is given by a CRS-neoclassical production function satisfying the Inada conditions, now with Harrod-neutral technological progress:

$$\tilde{Y} = F(K, TL) = F(\frac{K}{TL}, 1)TL \equiv f(\tilde{k})TL,$$

where $\tilde{k} \equiv K/(TL)$, and the technology level, T, grows exogenously acording to $T_t = T_0 e^{\gamma t}$, where $\gamma \geq 0$.Of course Let the real wage faced by the firm at time t be denoted w_t. From the firm's point of view, at every t, both w_t and K_t are given, and the firm chooses L_t so as to satisfy $\partial \tilde{Y}_t/\partial L_t = F_2(K_t, T_t L_t)T_t$ $= \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right] T_t \equiv \psi(\tilde{k}_t)T_t = w_t$. Consequently, $\psi(\tilde{k}_t) = w_t/T_t \equiv \tilde{w}_t$, so that $\tilde{k}_t = \psi^{-1}(\tilde{w}_t) \equiv \tilde{k}(\tilde{w}_t)$, $\tilde{k}' > 0$ in analogy with (14.25) above. Assume that also w_t grows at the rate γ. Then $\tilde{w}_t$ will be a constant, $\tilde{w}$. The chosen effective capital-labor ratio, $\tilde{k}_t$, will likewise be a constant, $\tilde{k}(\tilde{w})$. This requires L_t $= K_t/(T_t\tilde{k}_t) = K_t/(T_t\tilde{k}(\tilde{w})) \equiv \hat{k}_t/\tilde{k}(\tilde{w})$ and implies $F_K(K_t, T_t L_t) = f'(\tilde{k}(\tilde{w}))$, a constant.

Thus, by (14.31) together with (i) of Lemma 2,

$$\dot{q}_t = (r + \delta)q_t - f'(\tilde{k}(\tilde{w})) - \frac{(q_t - 1)^2}{2\beta}. \tag{**}$$

We assume that $f'(\tilde{k}(\tilde{w})) > r + \delta$. Then, for some $q > 1$, $f'(\tilde{k}(\tilde{w})) = (r + \delta)q$, and for some unique $q = q^*$ even larger. $\dot{q}_t = 0$.

(to be continued)

14.2 Marginal q and average q

Our q above, determining investment, should be distinguished from what is usually called Tobin's q or average q. Let p_{It} denote the current purchase price (relative to some output price index) per unit of the investment good (before installation). Then *Tobin's* q, or *average* q, q_t^a, is defined as $q_t^a \equiv V_t/(p_{It}K_t)$ (the top index "a" stands for "average"). Tobin's q is thus the ratio of the market value of the firm to the replacement value (before installation costs) of the firm's capital stock. In our simplified context we have $p_{It} \equiv 1$ (the price of the investment good is the same as that of the output good). In this case Tobin's q

is

$$q_t^a = \frac{V_t}{K_t} = \frac{V^*(K_t, t)}{K_t}, \tag{14.32}$$

where the second equality, by definition of the value function $V^*(K_t, t)$, holds for an optimizing firm.

Conceptually, q_t^a is different from the firm's shadow price on capital, our q_t in the previous sections. In the language of the q-theory of investment, this q_t is called the "marginal q". This name is natural since along the optimal path we have $q_t = \partial V^*(K_t, t)/\partial K_t$ according to (14.20). Letting q_t^m be our symbol for "marginal q", we can thus, for the case $p_{It} \equiv 1$, write

$$q_t^m = q_t = \frac{\partial V^*(K_t, t)}{\partial K_t}. \tag{14.33}$$

If we want to allow for $p_{It} \neq 1$, we define "marginal q" as representing the value to the optimizing firm of one *extra* unit of installed capital *relative* to the price of the investment good, that is, $q_t^m \equiv q_t/p_{It} = (\partial V^*(K_t, t)/\partial K_t)/p_{It}$.

The two concepts, average q and marginal q, have not always been clearly distinguished in the literature. What is directly relevant to the investment decision is marginal q. The analysis above showed that optimal investment is an increasing function of q^m. Further, the analysis showed that a "critical" value of q^m is 1 in the sense that if and only if $q^m > 1$, is positive gross investment warranted.

The importance of the variable q^a from the point of view of the economic researcher is that it can be measured empirically as the firms' market value (the sum of equity and debt) relative to the replacement value of the firm's capital stock, i.e., excluding installation costs. Since q^m is much harder to measure than q^a, it is important to know the theoretical relationship between q^m and q^a. Fortunately, we have a simple theorem giving conditions under which $q^m = q^a$.

THEOREM (Hayashi, 1982) Assume that the firm is a price taker, that the production function F is neoclassical and concave in (K, L), and that the installation cost function G is convex in (I, K).[14] Then, along an optimal path we have:

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in (K, L) and/or G is strictly convex in (I, K).

Proof. See Appendix D.

The intuitive background to point (i) of the theorem is the following. Let the policy $(L_t, I_t)_{t=0}^{\infty}$ be the optimal policy for a firm with capital stock $K_0 > 0$ at

[14] That is, in addition to (14.3), we assume $G_{KK} \geq 0$ and $G_{II}G_{KK} - G_{IK}^2 \geq 0$. The specification in Example 1 above satisfies this.

time 0. Assume the production and installation cost functions are homogeneous of degree 1. Then also the cash flow $R(K, L, I) \equiv F(K, L) - G(I, K) - wL - I$ is homogeneous of degree 1. Let another firm with the same production and installation cost functions have capital stock $K_0' > 0$ at time 0. The policy $(L_t', I_t') = (L_t, I_t) \cdot K_0'/K_0$ for all $t \geq 0$ must then be optimal for this firm. Moreover, the firm's optimized market value must be K_0'/K_0 times the optimized market value of the first firm. Indeed, for all $t \geq 0$ we will have that the optimized forward-looking market value, $V^*(K_t, t)$, is proportional to the current capital stock K_t. The factor of proportionality is what we have denoted q_t^a. The in this way ascertained linear relationship $V^*(K_t, t) = q_t^a \cdot K_t$ implies that $q_t^m \equiv \partial V^*(K_t, t)/\partial K_t = q_t^a$ for all $t \geq 0$.

The assumption that the firm is a price taker may, of course, seem critical. The Hayashi theorem has been generalized, however. Also a monopolistic firm, facing a downward-sloping demand curve and setting its own price, may have a cash flow which is homogeneous of degree one in the three variables K, L, and I. If so, then the condition $q_t^m = q_t^a$ for all $t \geq 0$ still holds (Abel 1990). Abel and Eberly (1994) show that .... NN present further generalizations.

In any case, when q^m is approximately equal to (or just proportional to) q^a, the theory gives a remarkably simple operational investment function,

$$I = m(q^a)K,$$

cf. (14.28). At the macro level we interpret q^a as the market valuation of the mass of firms relative to the replacement value of their total capital stock. Under the conditions in (i) of the Hayashi theorem, the market valuation also indicates the *marginal* earnings potential of the firms, hence, it becomes a determinant of their investment. This establishment of a relationship between the stock market and firms' aggregate investment is the basic point in Tobin (1969).

Does the hypothesis of homogeneity of degree one of the cash flow in K, L, and I make economic sense? According to a standard replication argument it does. Suppose a given firm has K units of installed capital and produces Y units of output with L units of labor. When at the same time the firm invests I units of account in new capital, it obtains the cash flow R after deducting the installation costs, $G(I, K)$. Then it makes sense to assume that the firm could do the same thing at another place, hereby doubling its cash-flow.

14.3 Applications

14.3.1 Capital installation costs in a closed economy

Allowing for convex capital installation costs in the economy has far-reaching implications for the causal structure of a model of a closed economy. Investment decisions attain an active role in the economy and forward-looking expectations become important for these decisions. Expected future market conditions and announced future changes in corporate taxes and depreciation allowance will affect firms' investment already today.

The essence of the matter is that current and expected future interest rates have to adjust for aggregate saving to equal aggregate investment, that is, for the output market to clear. Given full employment ($L_t = \bar{L}_t$), the output market clears when aggregate supply equals aggregate demand, i.e.,

$$F(K_t, \bar{L}_t) - G(I_t, K_t) \text{ } (= \text{value added} \equiv GDP_t) = C_t + I_t,$$

where C_t is determined by the intertemporal utility maximization of the forward-looking households, and I_t is determined by the intertemporal value maximization of the forward-looking firms facing strictly convex installation costs. Like in the determination of C_t, current and expected future interest rates now also matter for the determination of I_t. This is the first time in this book where *clearing in the output market* is assigned an *active* role. In the earlier models investment was just a passive reflection of household saving. Desired investment was automatically equal to the residual of national income left over after consumption decisions had taken place. Nothing had to adjust to clear the output market, neither interest rates nor output. In contrast, in the present framework adjustments in interest rates and/or the output level are needed for the continuous clearing in the output market and these adjustments are decisive for the macroeconomic dynamics.

A related implication of the theory is that we have to discard the simple conception from our previous models that the real interest rate is the variable which adjusts so as to clear a rental market for capital goods. The interest rate will no longer be tied down by a requirement that such markets clear, and will, even under perfect competition, no longer in equilibrium equal the net marginal productivity of capital. This is seen for instance in the formula (14.15).

In actual economies there may of course exist "secondary markets" for used capital goods and markets for renting capital goods owned by others. In view of installation costs and similar, however, shifting capital goods from one plant to another is generally costly. Therefore the turnover in that kind of markets tends to be limited (with the exception of rental markets for cars, trucks, air planes, and similar). And, importantly for our theory, the effective capital cost per time unit for a firm that hire its capital goods, rather than buying them, will still consist not only of the simple rental rate (interest plus depreciation costs, $r + \delta$) but also costs associated with installation and presumably also the later dismantling.

In for instance Abel and Blanchard (1983), a Ramsey-style model integrating the q-theory of investment and Hayashi's theorem is presented. The authors study the two-dimensional general equilibrium dynamics resulting from the adjustment of current and expected future interest rates *needed for the output market to clear*. Adjustments of the whole structure of interest rates (the yield curve) take place and constitute the equilibrating mechanism in the output and asset markets.

By having output market equilibrium playing this role in the model, a first step is taken towards medium- and short-run macroeconomic theory. We take further steps in later chapters, by allowing imperfect competition and nominal price rigidities to enter the picture. Then the demand side gets an active role both in the determination of q (and thereby investment) and in the determination of aggregate output and employment. This is what Keynesian theory (old and new) deals with.

In the remainder of this chapter we will still assume perfect competition in all markets including the labor market. In this sense we will stay within the neoclassical framework (supply-dominated models) where, by instantaneous adjustment of the real wage, labor demand continuously matches labor supply. The next two subsections present simple examples of how Tobin's q-theory of investment can be integrated into the neoclassical framework. To avoid the complications arising from an endogenous interest rate, the focus is on a small open economy. In that context, households financial wealth is distinct from the market value of the capital stock. Anyway, our focus will be on firms' capital accumulation, and the analysis will largely not need appeal to Hayashi's theorem.

14.3.2 A small open economy with capital installation costs

By introducing convex capital installation costs in a model of a small open economy (SOE), we avoid the counterfactual outcome that the capital stock adjusts *instantaneously* when the interest rate in the world financial market changes. In simple neoclassical models for a small open economy, without convex capital installation costs, a rise in the interest rate leads immediately to a complete adjustment of the capital stock so as to equalize the net marginal productivity of capital to the new higher interest rate. Moreover, in that model expected *future* changes in the interest rate or in corporate taxes and depreciation allowances do *not* trigger an investment response until these changes actually happen. In contrast, when convex installation costs are present, expected future changes tend to influence firms' investment already today.

We assume:

1. Perfect mobility across borders of goods and financial capital.
2. Domestic and foreign financial claims are perfect substitutes.
3. No mobility across borders of labor.
4. Labor supply is inelastic.
5. Both the production function and the capital installation cost function are homogeneous of degree 1.

In this setting the SOE faces an exogenous interest rate, r, given from the world financial market. We assume r is a positive constant. There are N firms, $i = 1, 2, \ldots, N$, all facing the same production function, $F(K, L)$, which is neoclassical, has CRS, and satisfies the Inada conditions. The firms also face the same installation cost function, $G(I, K)$, which is homogeneous of degree 1. Markets are competitive, and firms have perfect foresight and maximize profits intertemporally.

We consider two cases.

Case 1: Constant labor supply and absence of technical change

Let $\bar{L} > 0$ denote the constant labor supply. At time t firm i has capital equal to K_{it} (predetermined due to the convex installation costs) and adjusts its employment L_{it} so as to satisfy

$$F_L(K_{it}, L_{it}) = F_L(k_{it}, 1) = w_t, \qquad i = 1, 2, \ldots, N, \tag{14.34}$$

where $k_{it} \equiv K_{it}/L_{it}$ and w_t is the current market wage. The first equality follows from Euler's theorem saying that if F is homogeneous of degree 1, then the partial derivatives of F are homogeneous of degree 0. We see that the chosen capital-labor ratio, k_{it}, will be the same for all firms and thus the same as the aggregate capital-labor ratio, K_t/L_t, where $K_t \equiv \sum_{i=1}^{N} K_{it}$ and $L_t \equiv \sum_{i=1}^{N} L_{it}$. Clearing in the labor market requires that $L_t = \bar{L}$. Hence,

$$k_{it} = K_t/\bar{L} \equiv k_t, \qquad i = 1, 2, \ldots, N.$$

Substituting this into (14.34), and rearranging, we have, for all $t \geq 0$,

$$w_t = F_L(K_t, \bar{L}) = F_L(k_t, 1) = f(k_t) - k_t f'(k_t) \equiv w(k_t), \qquad w' = -kf'' > 0, \tag{14.35}$$

where we have introduced the production function in intensive form.

On the one hand, since k_t is predetermined, the equation (14.35) *determines* the market real wage w_t. On the other hand, the labor input chosen by firm i, with given K_{it} and facing a given market wage rate w_t, will be

$$L_{it} = K_{it}/k_t = K_{it}/w^{-1}(w_t), \tag{14.36}$$

where the inverse function $w^{-1}(w_t)$ comes from (14.35). Looking ahead, the firm knows that at time s in the future, depending on its expected w_s and planned capital, K_{is}, its optimal employment will be $L_{is} = K_{is}/w^{-1}(w_s)$.

We now put ourselves in the position of firm i and try to make an optimal production and investment plan for this firm as seen from time 0. To pin down the optimal plan, given r and the expected evolution of the market wage rate, $(w_t)_{t=0}^{\infty}$, we derive from the first-order conditions in Section 14.1.1 two coupled differential equations in K_{it} and q_{it}. By (14.28),

$$\dot{K}_{it} = I_{it} - \delta K_{it} = (m(q_{it}) - \delta)K_{it}, \qquad K_{i0} > 0 \text{ given.} \tag{14.37}$$

Since the capital installation cost function G is homogeneous of degree 1, point (iv) of Lemma 1 applies, and so we can write (14.31) as

$$\begin{aligned}\dot{q}_{it} &= (r+\delta)q_{it} - F_K(K_{it}, K_{it}/w^{-1}(w_t)) + g(m(q_{it})) - (q_{it}-1)m(q_{it}) \\ &\equiv (r+\delta)q_{it} - F_K(1, 1/w^{-1}(w_t)) + g(m(q_{it})) - (q_{it}-1)m(q_{it}), \end{aligned} \tag{14.38}$$

where we have first applied (14.36) and then again Euler's theorem on F_K.

As r and w_t are exogenous to the firm, the planned capital stock, K_{it}, and its shadow price, q_{it}, are the only endogenous variables in the differential equations (14.39) and (14.40). In addition, we have an initial condition for K_i and a necessary transversality condition involving q_i, namely

$$\lim_{t\to\infty} K_{it}q_{it}e^{-rt} = 0.$$

From the firm's perspective, the problem is to find out what its optimal investment "guide", q_{i0}, is. If this problem is solved, the dynamic system (14.37)-(14.38) determines the capital accumulation of firm i. Indeed, given both q_{i0} and K_{i0}, (14.37) determines the change in K_i in the short time interval $(0, \varepsilon)$ and thereby the new K_i at time ε. And given both q_{i0} and, from the market, w_0, (14.38) similarly determines the new q_i at time ε, and so on.

We see that the firm's optimal path depends on the evolution of w_t, which according to (14.35) is determined by the evolution of aggregate capital, K_t. This illustrates the importance of general equilibrium analysis. The parts depends on the system as a whole, and the whole depends, of cource, on its parts.

We need to find out how aggregate capital will move. This is a complicated matter, unless the firms have the same initial optimal q. This requires their K_{i0}

to be the same, which we will now assume they are. Consequently, $K_{it} = K_t/N$ and $q_{it} = q_t$ for all $t \geq 0$. Then (14.37) implies

$$\dot{K}_t = I_t - \delta K_t = (m(q_t) - \delta)K_t, \qquad K_0 > 0 \text{ given.} \tag{14.39}$$

Moreover, since $K_t/\bar{L} \equiv k_t = w^{-1}(w_t)$, we have, again applying Euler's theorem,

$$F_K(1, 1/w^{-1}(w_t)) = F_K(1, \bar{L}/K_t) = F_K(K_t, \bar{L}).$$

Hence, with $q_{it} = q_t$ for all $t \geq 0$, (14.38) becomes

$$\dot{q}_t = (r + \delta)q_t - F_K(K_t, \bar{L}) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{14.40}$$

Finally, the above transversality condition implies

$$\lim_{t\to\infty} K_t q_t e^{-rt} = 0. \tag{14.41}$$

Fig. 14.4 shows the phase diagram for these two coupled differential equations. Let q^* be defined as the value of q satisfying the equation $m(q) = \delta$. Since $m' > 0$, q^* is unique. Suppressing for convenience the explicit time subscripts, we then have

$$\dot{K} = 0 \text{ for } m(q) = \delta, \text{ i.e., for } q = q^*.$$

As $\delta > 0$, we have $q^* > 1$. This is so because also mere reinvestment to offset capital depreciation requires an incentive, namely that the marginal value to the firm of replacing worn-out capital is larger than the purchase price of the investment good (since the installation cost must also be compensated). From (14.39) is seen that

$$\dot{K} \gtrless 0 \text{ for } m(q) \gtrless \delta, \text{ respectively, i.e., for } q \gtrless q^*, \text{ respectively,}$$

cf. the horizontal arrows in Fig. 14.4.

From (14.40) we have

$$\dot{q} = 0 \text{ for } 0 = (r + \delta)q - F_K(K, \bar{L}) + g(m(q)) - (q - 1)m(q). \tag{14.42}$$

If, in addition $\dot{K} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta$), this gives

$$0 = (r + \delta)q^* - F_K(K, \bar{L}) + g(\delta) - (q^* - 1)\delta, \tag{14.43}$$

where the right-hand-side is increasing in K, in view of $F_{KK} < 0$. Hence, there exists at most one value of K such that the steady state condition (14.43) is satisfied. And our assumption that F satisfies the Inada conditions ensures that

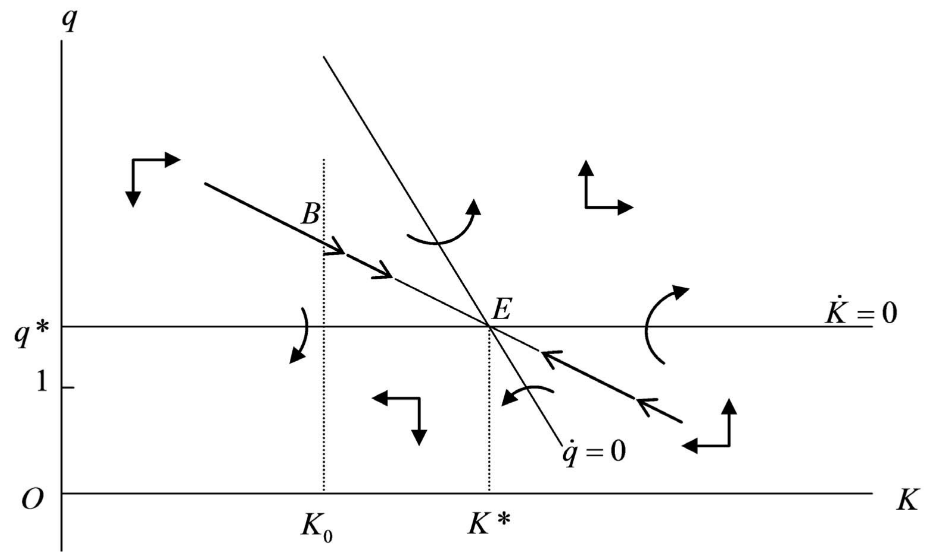

Figure 14.4: Phase diagram for investment dynamics in a small open economy (a case where $\delta > 0$).

such a value exists since (14.43) gives $F_K(K, \bar{L}) = rq^* + g(\delta) + \delta > 0$. This value is denoted K^*, cf. the steady state point E in Fig. 14.4.

The next question is: what is the slope of the $\dot{q} = 0$ locus? In Appendix E, calculating dq/dK subject to the condition (14.42), we find that at least in a neighborhood of the steady-state point E, this slope is negative in view of the assumption $r > 0$ and $F_{KK} < 0$. From (14.40) we see that

$$\dot{q} \lesseqgtr 0 \text{ for points to the left and to the right, respectively, of the } \dot{q} = 0 \text{ locus,}$$

since $F_{KK}(K_t, \bar{L}) < 0$. The vertical arrows in Fig. 14.4 show these directions of movement.

Altogether the arrows in the four regions, I, II, III, and IV, in the phase diagram show that the steady state E is a saddle point. If we imagine that instead of aggregate capital, K, along the horizontal axis in Fig. 14.4, we have K/N and the scale is correspondingly adjusted by $1 : N$, then the phase diagram depicts the dynamics of the individual firm. Hence, from now, we can interpret the figure as describing the dynamics of a "representative firm" as well as the economy as a whole.

As the capital stock is pre-determined, initially the economy must be situated at some point on the vertical line $K = K_0$ in Fig. 14.4. A reasonable supposition is that the initial value of the jump variable q will be the ordinate to the point of intersection of this line and the saddle path. Over time the economy will then move along the saddle path towards the steady state. Along this path the transversality condition (14.41) is satisfied since $r > 0$. From the perspective of the alike firms, following this path means intertemporal profit maximization under perfect foresigt. This is implied by Mangasarian's sufficient conditions, as we saw at the end of Section 14.1.1. And by construction, there is clearing in the labor market for all t along the path. So we have found an equilibrium path in this economy.

Uniqueness?* Could there exist other equilibrium paths? The fact that the Hamiltonian is a concave function of (K, L, I) and that there is a unique steady state makes uniqueness "likely" to hold. All other paths in the phase diagram consistent with the model's differential equations can then "normally" be shown to violate a necessary transversality condition or some other condition that an equilibrium path with perfect foresight must satisfy. Let us check here.

Paths starting *above* the point B on the vertical line $K = K_0$ end up in Region II in Fig. 14.4. So both q and K are ultimately growing for all future t, and the transversality condition (14.41) can be shown to be violated (see Appendix F). Hence such a path cannot be optimal. A path starting *below* the point B on the vertical line $K = K_0$ will ultimately have both K and q declining. Since neither K nor q can become negative,[15] such a path will *not* violate the transversality condition. Yet our intuitive feeling is that such a path cannot be optimal. And by appealing to Hayashi's theorem we can prove this. Indeed, from this theorem we know that the homogeneity of both F and G implies that the maximized value of the firm satisfies

$$V^*(K_0, 0) = q_0 K_0,$$

where q_0 is the largest possible initial value of the shadow price, given the requirement that the associated trajectory in the phase diagram does not violate the transversality condition (14.41). The largest possible initial value with this property is q_B. If $q_0 < q_B$ we have $q_0 K_0 < q_B K_0$. Thus, $V^*(K_0, 0) = q_B K_0$. The argument is the same if we start from a $K_0 > K^*$.

We conclude that the trajectory starting at (K_0, q_B) and moving along the saddle path towards the steady state E in Fig. 14.4 is the unique equilibrium path of the model.

The effect of an unanticipated rise in the interest rate Suppose that until time 0 the economy has been in the steady state E in Fig. 14.4. Then, an unexpected shift in the interest rate occurs so that the new interest rate is a constant $r' > r$. We assume that the new interest rate is rightly expected to remain at this level forever. From (14.39) we see that the $\dot{K} = 0$ locus, hence

[15] Free disposal rules out a negative shadow price, cf. Section 14.1.2.

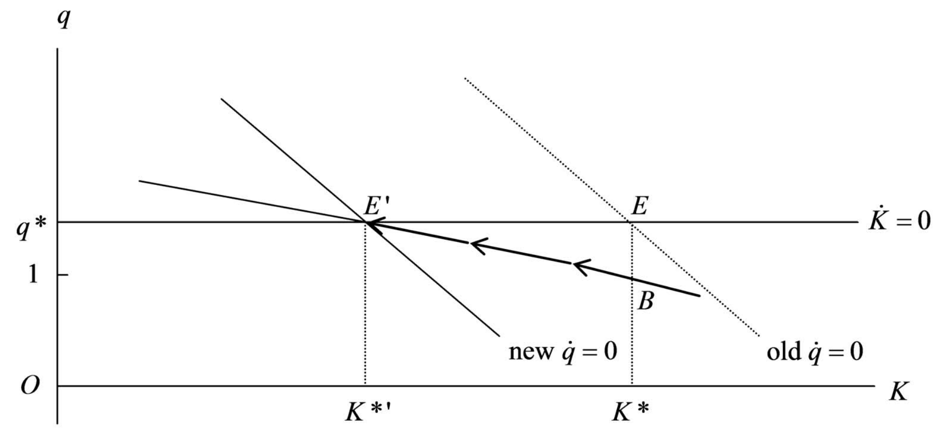

Figure 14.5: Phase portrait of an unanticipated rise in r (the case $\delta > 0$).

also q^*, is unaffected by this shift. However, (14.42) implies that the $\dot{q} = 0$ locus, and so also K^*, shift to the left, in view of $F_{KK}(K, \bar{L}) < 0$.

Fig. 14.5 illustrates the situation for $t > 0$. At time $t = 0$ the shadow price q jumps down to a level corresponding to the point B in Fig. 14.5. There is now a heavier discounting of the future benefits that the marginal unit of capital can provide. As a result the incentive to invest is diminished and gross investment will not even compensate for the depreciation of capital. Hence, the capital stock decreases gradually. This is where we see a crucial role of convex capital installation costs in an open economy. For now, the installation costs are the costs associated with disinvestment (dismantling and selling out of machines). If these convex costs were not present, we would get the same counterfactual prediction as from the previous open-economy models in this book, namely that the new steady state is attained immediately after the shift in the interest rate.

As the capital stock is diminished, the marginal productivity of capital rises and so does q. The economy moves along the new saddle path and approaches the new steady state E' as time goes by.

Suppose the described decrease in the capital stock is not considered desirable from a social point of view. This could be because of positive external effects of investing and working with capital equipment, a kind of "learning by doing". Then the government could decide to implement an investment subsidy $\sigma \in (0, 1)$. Then, to attain the investment level I, purchasing the investment goods costs $(1 - \sigma)I$. Assuming the subsidy is financed by a tax not affecting firms' behavior, investment is increased again and the economy will over time end up at a steady-state level of K higher than without the subsidy. In Exercise 14.? the reader is asked to examine whether q^* is affected by such a policy.

The effect of an unanticipated one-time rise in the labor force Suppose that until time 0 the economy has been in the steady state E in Fig. 14.4. Then an unexpected sharp increase in the labor force to the level $\bar{L}' > \bar{L}$ occurs (a sudden immigration, say). Assume the new size of the labor force is rightly expected to remain at this level forever. While q^* remains unchanged, immediately after the labor supply shock, K^* shifts *to the right*, in view of steady-state condition (14.43) combined with $F_{KL}(K, \bar{L}) > 0$ (this complementarity follows from F being neoclassical with CRS). Hence, the $\dot{q} = 0$ locus shifts. A dynamics in the opposite direction compared with the above case with a positive permanent interest rate shock arrises, and the economy moves along the new saddle path and approaches a new steady state with a higher level of capital. The steady-state real wage was $w^* = F_L(K^*, \bar{L})$ in the old steady state and will be $w^{*\prime} = F_L(K^{*\prime}, \bar{L}')$ in the new. In view of (14.43), including the constant interest rate, $F_K(K^*, \bar{L})$ and $F_K(K^{*\prime}, \bar{L}')$ have to be the same. Because F_K is homogeneous of degree zero, the ratios $K^*/\bar{L}$ and $K^{*\prime}/\bar{L}'$ then have to be the same, and so this holds for w^* and $w^{*\prime}$ as well. The rise in the labor force thus leads in the medium term to a rise in capital so as to leave the capital-labor ratio and the real wage unchanged.

Case 2: A growing small open economy with capital installation costs*

The basic assumptions are the same as in the previous section except that now labor supply, $\bar{L}_t$, grows at the constant rate $n \geq 0$, while the technology level, T, grows at the constant rate $\gamma \geq 0$ (both rates exogenous and constant). The world market real interest rate, r, is still a constant and satisfies $r > \gamma + n$. We have full employment: $L_t = \bar{L}_t = \bar{L}_0 e^{nt}$. So

$$\tilde{Y} = F(K, T\bar{L}) = F(\frac{K}{T\bar{L}}, 1)T\bar{L} \equiv f(\tilde{k})T\bar{L},$$

where $\tilde{k} \equiv K/(T\bar{L})$ and f satisfies $f' > 0$ and $f'' < 0$. In view of perfect competition, the market-clearing real wage at time t is determined as

$$w_t = F_2(K_t, T_t\bar{L}_t)T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right] T_t \equiv \tilde{w}(\tilde{k}_t)T_t,$$

where both $\tilde{k}_t$ and T_t are predetermined. The equilibrium real wage at any time is thus determined by the pre-determined effective capital-labor ratio.

All firms are again assumed completely alike. By analogue logic as in Case 1 above, we can go directly to the aggregate dynamics. Log-differentiation of $\tilde{k} \equiv K/(T\bar{L})$ with respect to time gives $\dot{\tilde{k}}_t/\tilde{k}_t = \dot{K}_t/K_t - (\gamma + n)$. Substituting (14.39), we get

$$\dot{\tilde{k}}_t = [m(q_t) - (\delta + \gamma + n)]\, \tilde{k}_t. \tag{14.44}$$

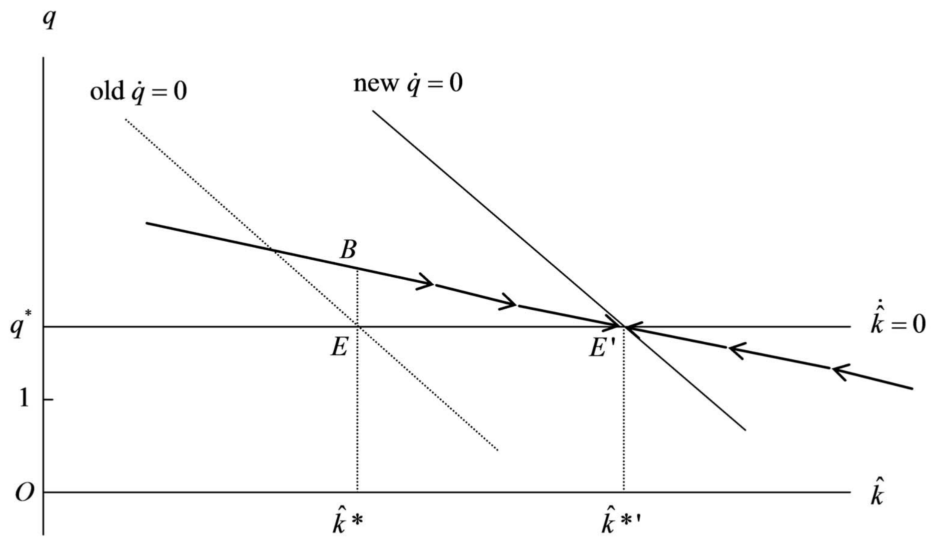

Figure 14.6: Phase portrait of an unanticipated fall in r (a growing economy with $\delta + \gamma + n \geq \gamma + n > 0$).

The change in the shadow price of installed capital is described by

$$\dot{q}_t = (r + \delta)q_t - f'(\tilde{k}_t) + g(m(q_t)) - (q_t - 1)m(q_t), \tag{14.45}$$

by straightforward generalization of (14.40). Finally, the transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t q_t e^{-(r-\gamma-n)t} = 0, \tag{14.46}$$

must hold.

The differential equations (14.44) and (14.45) constitute our new dynamic system. Fig. 14.6 shows the phase diagram, which is qualitatively similar to that in Fig. 14.4. We have

$$\dot{\tilde{k}} = 0 \text{ for } m(q) = \delta + \gamma + n, \text{ i.e., for } q = q^*,$$

where q^* is defined by the requirement $m(q^*) = \delta + \gamma + n$. Notice, that when $\gamma + n > 0$, we get a larger steady state value q^* than in the previous section. This is because now a higher investment-capital ratio is required for a steady state to be possible. In view of $r > \gamma + n$, the transversality condition (14.46) is satisfied in the steady state.

From (14.45) we see that $\dot{q} = 0$ now requires

$$0 = (r + \delta)q - f'(\tilde{k}) + g(m(q)) - (q - 1)m(q).$$

If, in addition $\dot{\tilde{k}} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta + \gamma + n$), this gives

$$0 = (r + \delta)q^* - f'(\tilde{k}) + g(\delta + \gamma + n) - (q^* - 1)(\delta + \gamma + n). \tag{14.47}$$

Here, the right-hand-side is increasing in $\tilde{k}$ (in view of $f''(\tilde{k}) < 0$). Hence, the steady-state value $\tilde{k}^*$ of the effective capital-labor ratio is unique, cf. the point E in Fig. 14.6.

By the assumption $r > \gamma + n$ we have, at least in a neighborhood of E in Fig. 14.6, that the $\dot{q} = 0$ locus is negatively sloped (see Appendix E).[16] Again the steady state is a saddle point, and the economy moves along the saddle path towards the steady state.

It is still true, as in the simple Case 1 above, that the market real wage per unit of effective labor plays a role for the firms' adjustment process. In the long run, however, it is the real wage that adjusts. Indeed, in the steady state we have $w_t^* = \tilde{w}(\tilde{k}^*)T_t$, where $\tilde{k}^*$ is determined by the production function, the installation cost function, and the parameters r, δ, γ, and n, cf. (14.47).

An unanticipated *fall* in r. Assume that until time 0, the economy has been in the steady state E. Then, an unexpected shift in the interest rate to a lower constant level, r', takes place. Assume the new interest rate is rightly expected to remain at this level forever. In view of $f'' < 0$, the lower interest rate shifts the $\dot{q} = 0$ locus to the right, as illustrated in Fig. 14.6. The shadow price, q, immediately jumps up to a level corresponding to the point B in the figure. The economy moves along the new saddle path and approaches the new steady state E' with a higher effective capital-labor ratio as time goes by. In Exercise 14.? the reader is asked to examine the analogue situation where an unanticipated downward shift in the rate of technological progress takes place.

14.4 Concluding remarks

Tobin's q-theory of investment gives a remarkably simple operational macroeconomic investment function, in which aggregate investment is an increasing function of the valuation of the firms by the stock market relative to the replacement value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

When faced with strictly convex installation costs, the firm has to take the *future* into account to invest optimally. Therefore, the firm's *expectations* become

[16] In our perfect foresight model we in fact *have* to assume $r > \gamma + n$ for the firm's maximization problem to be well-defined. If instead $r \leq \gamma + n$, the market value of the representative firm would be infinite, and maximization would loose its meaning.

important. Owing to the strictly convex installation costs, the firm adjusts its capital stock only *gradually* when new information arises.

By incorporating these features, Tobin's q-theory helps explaining the sluggishness in investment that corresponds to our intuition and which we see in the data. The theory avoids the unrealistic picture of firms' capital as a production factor that can move instantaneously across firms and industries. And it avoids the counterfactual outcome from earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world market changes. So the theory takes into account the time lags in capital adjustment in real life. Possibly, this feature can be abstracted from in long-run analysis and models of economic growth, but not in short- and medium-run analysis.

Many econometric tests of the q theory of investment have been made, often with critical implications. Movements in q^a, even taking account of changes in taxation, seems capable of explaining only a minor part of the movements in investment. And the estimated equations relating fixed capital investment to q^a typically give strong auto-correlation in the residuals. Other variables, in particular availability of current profits for internal financing, seem to have explanatory power independently of q^a (see Abel 1990, Chirinko 1993, Gilchrist and Himmelberg, 1995). There is thus reason to be sceptical towards the notion that *all* information of relevance for the investment decision is reflected by the market valuation of firms. The assumption in Hayashi's theorem (and its generalizations), that firms' cash flow tends to be homogeneous of degree one with respect to K, L, and I, may of course also be questioned.

Further circumstances are likely to relax the link between q^a and investment. In the real world with many production sectors, physical capital is heterogeneous. If for example a sharp unexpected rise in the price of energy takes place, a firm with energy-intensive technology is likely to loose in market value. At the same time it has an incentive to invest in energy-saving capital equipment. Hence, we might observe a fall in the firm's q^a at the same time as its investment increases.

Imperfections in credit markets are ignored by the q-theory. Their presence further loosens the relationship between q^a and investment and may help explain the observed positive correlation between investment and corporate profits.

It could also be questioned that capital installation costs really have the hypothesized *strictly convex* form. It is one thing that there are costs associated with installation, reorganizing and retraining etc., when new capital equipment is procured. But should we expect these costs to always be strictly convex in the volume of investment? To think about this, let us for a moment ignore the role of the existing capital stock and write total installation costs as $J = G(I)$ with $G(0) = 0$. It does not seem problematic to assume $G'(I) > 0$ for $I > 0$.

The question concerns the assumption that $G''(I) > 0$ at *all* levels of I and thereby that the average installation cost $G(I)/I$ is increasing in I *everywhere*.[17] Indeed, capital installation may involve indivisibilities and fixed costs, in which case a tendency to *decreasing* average costs arises. So, at least at firm level there may be reason to expect unevenness in capital adjustment rather than the above smooth adjustment.

Because of the mixed empirical success of the convex installation cost idea, additional factors have been introduced to account for sluggish and sometimes lumpy capital adjustment: uncertainty, investment irreversibility and option values, indivisibility, and financial frictions due to bankruptcy costs. In their book, *Investment Under Uncertainty,* Princeton University Press, 1994, Dixit and Pindyck show that the traditional present value maximization rule for capital investment can lead to wrong answers because it ignores the irreversibility of many investment decisions, hence the importance of the option of delaying an investment. A survey of the theory and empirics about fixed capital investment is given in Caballero (1999).

The different approaches may be complementary rather than substitutes. It turns out that the q-theory of investment has recently been somewhat rehabilitated from both a theoretical and an empirical point of view. At the theoretical level Wang and Wen (2010) show that financial frictions in the form of collateralized borrowing at the firm level can give rise to strictly convex adjustment costs at the aggregate level yet at the same time generate lumpiness in plant-level investment. For large firms, unlikely to be much affected by financial frictions, Eberly et al. (2008) find that the q-theory does a quite good job in explaining investment behavior.

Whatever the detailed merits or weaknesses of the q-theory of investment, its basic point, that capital adjustment is time-consuming and involves adjustment costs, remains in force. Varieties of the q-theory of investment are widely used in short- and medium-run macroeconomics, both because of the simplicity of the theory and the link it establishes between asset markets and firms' investment. Elements of the q-theory have also had an important role in studies of housing market dynamics, a theme to which we return in the next chapter.

14.5 Literature notes

The label "Tobin's q-theory of investment is a short-hand for a fusion of two strands of contributions to macroeconomic investment theory. One is the convex

[17] Indeed, for $I \neq 0$ we have $d[G(I)/I]/dI = [IG'(I) - G(I)]/I^2 > 0$, when G is strictly convex ($G'' > 0$) and $G(0) = 0$.

adjustment cost approach, originally developed by Lucas (1967), Gould (1968), Uzawa (1969), and Treadway (1969). Along this strand Mussa (1977) compares different ways of modeling internal and external adjustment costs and discusses conditions for aggregation. An early and instructive survey of the theory and empirics of firms' fixed capital investment is provided by Nickell (1979). The other strand is the hypothesis put forward in Tobin (1969) that firms' fixed capital investment is positively related to "average q". In fact this notion can be traced back to Keynes (1936, p. 151).

Later advances in the theory took place through the "synthesizing" contributions of Abel (1982).and Hayashi (1982). By distinguishing between "marginal q" and "average q", a synthesis of the convex adjustment cost approach and Tobin's macro-oriented "average q" approach was built, as surveyed in Abel (1990). For instance, Summers (1981) and Dixit (1990) use the framework to study dynamic effects of tax policy on corporate investment. From the empirical side, Abel and Blanchard (1986) computed series of expected present values of marginal profits based on US data. They found that the variations in this present values series are, surprisingly, due more to variations in the cost of capital than to variations in marginal profits. The present value series, although significantly related to investment, still left unexplained a large serially correlated fraction of the movement in investment. See also Blanchard, Rhee, and Summers (1993).

Tobin's q-theory of investment has been integrated in a variety of dynamic macroeconomic models. Abel and Blanchard (1983) studied dynamic effects of fiscal policy (an investment tax credit and taxes on output, profit, consumption, respectively) in a Ramsey model extended by adding strictly convex capital installation costs. In a similar framework Lim and Weil (2003) study effects of population aging on the stock market. ..... For instance Baxter and Cruzini (1993) and Christiano and Fisher (2003) integrate Tobin's q-theory into the RBC approach to business cycles. Groth and Madsen (2016) study medium-term fluctuations in a closed economy, arising in a Tobin's q framework combined with sluggishness in real wage adjustments.

More recent advances in the theory of lumpy capital investment are treated in, e.g., Zeira (1987), Dixit and Pindyck (1994), ... and Cooper (2003) and surveyed in Caballero (1999).

The theory and empirics concerning *inventory investment* also emphasize convex adjustment costs. Ramey and West (Handbook of Macro, vol. 1B, 1999) provide a survey.

14.6 Appendix

A. When value maximization is – and is not – equivalent to continuous static profit maximization

For the idealized case where tax distortions, asymmetric information, and problems with enforceability of financial contracts are absent, the Modigliani-Miller theorem (Modigliani and Miller, 1958) says that the market value (debt plus equity) does not depend on the level of the debt. The financial structure of the firm will be both indeterminate and irrelevant for production outcomes. Considering the firm described in Section 14.1, the implied separation of the financing decision from the production and investment decision can be exposed in the following way.

The Modigliani-Miller theorem in action Although the theorem allows for risk, we here ignore risk. Let the real debt of the firm be denoted B_t and the real dividends, X_t. We then have the accounting relationship

$$\dot{B}_t = X_t - (F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t).$$

A positive X_t represents dividends in the usual meaning (payout to the owners of the firm), whereas a negative X_t can be interpreted as emission of new shares of stock. Since we assume perfect competition, the time path of w_t and r_t is exogenous to the firm.

Consider first the firm's combined financing and production-investment problem, which we call *Problem I.* Assume (realistically) that those who own the firm at time 0 want it to maximize its net worth, i.e., the present value of expected future dividends:

$$\begin{aligned}
\max_{(L_t, I_t, X_t)_{t=0}^{\infty}} \tilde{V}_0 &= \int_0^\infty X_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.} \\
L_t &\geq 0, I_t \text{ "free"}, \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \quad K_t \geq 0 \text{ for all } t, \\
\dot{B}_t &= X_t - (F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t), \\
& \quad \text{where } B_0 \text{ is given}, \qquad (14.48) \\
\lim_{t\to\infty} B_t e^{-\int_0^t r_s ds} &\leq 0. \qquad \text{(NPG)}
\end{aligned}$$

The last constraint is the firm's No-Ponzi-Game condition, saying that a positive debt should in the long run at most grow at a rate which is *less* than the interest rate.

In Section 14.1 we considered another problem, namely a separate investment-production problem, which we name *Problem II*:

$$\begin{aligned}
\max_{(L_t,I_t)_{t=0}^{\infty}} V_0 &= \int_0^{\infty} R_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.,} \\
R_t &\equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \\
L_t &\geq 0, I_t \text{ free}, \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \;\; K_t \geq 0 \text{ for all } t.
\end{aligned}$$

In this problem the financing aspects are ignored. Regarding the relationship between Problem I and Problem II the following mathematical fact is useful.

LEMMA A1 Consider a continuous function $a(t)$ and a differentiable function $f(t)$. Then

$$\int_{t_0}^{t_1} (f'(t) - a(t)f(t))e^{-\int_{t_0}^{t} a(s)ds} dt = f(t_1)e^{-\int_{t_0}^{t_1} a(s)ds} - f(t_0).$$

Proof. By integration by parts from time t_0 to time t_1, we have

$$\int_{t_0}^{t_1} f'(t)e^{-\int_{t_0}^{t} a(s)ds} dt = f(t)e^{-\int_{t_0}^{t} a(s)ds} \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} f(t)a(t)e^{-\int_{t_0}^{t} a(s)ds} dt.$$

Hence,

$$\begin{aligned}
&\int_{t_0}^{t_1} (f'(t) - a(t)f(t))e^{-\int_{t_0}^{t} a(s)ds} dt \\
&= f(t_1)e^{-\int_{t_0}^{t_1} a(s)ds} - f(t_0). \qquad \square
\end{aligned}$$

CLAIM 1 If $(K_t^*, B_t^*, L_t^*, I_t^*, X_t^*)_{t=0}^{\infty}$ is a solution to Problem I, then $(K_t^*, L_t^*, I_t^*)_{t=0}^{\infty}$ is a solution to Problem II.

Proof. By (14.48) and the definition of R_t, $X_t = R_t + \dot{B}_t - r_t B_t$ so that

$$\tilde{V}_0 = \int_0^{\infty} X_t e^{-\int_0^t r_s ds} dt = V_0 + \int_0^{\infty} (\dot{B}_t - r_t B_t)e^{-\int_0^t r_s ds} dt. \tag{14.49}$$

In Lemma A1, let $f(t) = B_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty} \int_0^T (\dot{B}_t - r_t B_t)e^{-\int_0^t r_s ds} dt = \lim_{T\to\infty} B_T e^{-\int_0^T r_s ds} - B_0 \leq -B_0,$$

where the weak inequality is due to (NPG). Substituting this into (14.49), we see that maximum of net worth $\tilde{V}_0$ is obtained by maximizing V_0 and ensuring $\lim_{T\to\infty} B_T \exp(-\int_0^T r_s ds) = 0$, in which case net worth equals (maximized V_0) $-$ B_0, where B_0 is given. So a plan that maximizes net worth of the firm must also maximize V_0 in Problem II. □

In view of Claim 1, it does not matter for the firm's production and investment decision whether the investment is financed by issuing new debt or by issuing shares of stock. Moreover, if we assume investors do not care about whether they receive the firm's earnings in the form of dividends or valuation gains on the shares, the firm's dividend policy is also irrelevant. Hence, from now on we can concentrate on the investment-production problem, Problem II above.

The case with no capital installation costs Let (pure) profit at time t be denoted Π_t. Then:

$$\Pi_t = F(K_t, L_t) - w_t L_t - (r_t + \delta)K_t \equiv \Pi(K_t, L_t).$$

CLAIM 2 When there are no capital installation costs, Problem II can be reduced to a series of isolated static profit maximization problems.

Proof. Consider Problem II above with $G(I_t, K_t) \equiv 0$. Applying the Maximum principle, for every $t \geq 0$ we have the first-order conditions:

$$\begin{aligned} \partial H/\partial L_t &= F_L(K_t, L_t) - w_t = 0, && (*) \\ \partial H/\partial I_t &= -1 + q_t = 0, && (**) \\ \partial H/\partial K_t &= F_K(K_t, L_t) - q_t\delta = -\dot{q}_t + r_t q_t. && (***) \end{aligned}$$

(**) and (***) in combination implies $F_K(K_t, L_t) = r_t + \delta$. This condition and (*) make up the standard first-order conditions for static maximization of the profit Π_t as defined above. □

The background for this result is the following. In the absence of capital installation costs, the cash flow R_t can be written

$$R_t = F(K_t, L_t) - w_t L_t - I_t = \Pi_t + (r_t + \delta)K_t - (\dot{K}_t + \delta K_t), \tag{14.50}$$

since $I_t = \dot{K}_t + \delta K_t$. Hence,

$$V_0 = \int_0^\infty \Pi_t e^{-\int_0^t r_s ds} dt + \int_0^\infty (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt. \tag{14.51}$$

The first integral on the right-hand side of this expression is *independent* of the second. Indeed, in the absence of installation costs, the factor cost of production is $w_t L_t + (r_t + \delta)K_t$, since the operating cost of capital is $r_t + \delta$ per unit of capital service, whether the firm itself owns the used capital or rents it. can maximize the first integral by *renting* capital and labor, K_t and L_t, at the going factor prices, $r_t + \delta$ and w_t, respectively, such that $\Pi_t = \Pi(K_t, L_t)$ is maximized at each t.[18] The factor costs are accounted for in the definition of the profit function Π.

But maximization of V_0 requires maximization not only of the first integral in (14.51), but also the second, which can be interpreted as the present value of net revenue from accumulating capital and renting it out to others. In Lemma A1, let $f(t) = K_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty} \int_0^T (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt = K_0 - \lim_{T\to\infty} K_T e^{-\int_0^T r_s ds}. \qquad (14.52)$$

We see that maximization of the second integral in (14.51) requires, since K_0 is given, minimization of $\lim_{T\to\infty} K_T \exp(-\int_0^T r_s ds)$. This latter expression is always non-negative and can, when $r > 0$, be made zero by letting the long-run growth rate of K_t be less than the interest rate in the long run. (This reflects the principle expressed in Claim 3 below.) So the maximized value of the left-hand side of (14.52) is K_0. Substituting this into (14.51), we get $V_0 = \int_0^\infty \Pi_t \exp(-\int_0^T r_s ds) dt + K_0$.

The conclusion is that, given K_0,[19] V_0 is maximized if and only if K_t and L_t are at each t chosen such that $\Pi_t = \Pi(K_t, L_t)$ is maximized.

CLAIM 3 At least in the absence of capital installation costs, it does not pay the firm to accumulate costly capital in the long run at a rate as high as, or even higher than, the interest rate.

Proof. Sustained accumulation at a rate at least as high as the interest rate implies $\lim_{T\to\infty} K_T \exp(-\int_0^T r_s ds) > 0$. This inequality shows that the right-hand side of (14.52) is not maximized, and so V_0 is not maximized. □

[18] Continuously renting capital is from an accounting point of view equivalent to continuous renewal of short-term loans.

[19] Note that in the absence of capital installation costs, the historically given K_0 is no more "given" than the firm may instantly let it jump to a lower or higher level. In the first case the firm would immediately sell or rent out a bunch of its machines and in the latter case it would immediately buy or rent a bunch of machines. Indeed, without convex capital installation costs nothing rules out jumps in the capital stock at firm level. Such a jump will just leave the net worth of the firm unchanged, being counterbalanced by an immediate jump, in the opposite direction, of another asset in the firm's balance sheet.

The case with strictly convex capital installation costs Now we reintroduce the capital installation cost function $G(I_t, K_t)$, satisfying in particular the strict convexity assumption $G_{II}(I, K) > 0$ for all (I, K). Then we define current (pure) profit as

$$\tilde{\Pi}_t = F(K_t, L_t) - G(I_t, K_t) - w_t L_t - (r_t + \delta)K_t \equiv \tilde{\Pi}(K_t, L_t, I_t),$$

and write cash flow as

$$R_t = F(K_t, L_t) - G(I_t, K_t) - w_t L_t - (\dot{K}_t + \delta K_t) = \tilde{\Pi}_t + (r_t + \delta)K_t - (\dot{K}_t + \delta K_t).$$

Hence,

$$V_0 = \int_0^\infty \tilde{\Pi}_t e^{-\int_0^t r_s ds} dt + \int_0^\infty (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt.$$

From an accounting point of view this expression looks similar to (14.51). The new feature is, however, that the first integral on the right-hand side is no longer independent of the second integral. Via the installation costs, the current capital stock, K_t, and investment rate, I_t, affect both current profit and profit in the next instant.

As shown in the text, the intertemporally profit-maximizing firm will then adjust to a change in its environment, say a downward shift in r, by a *gradual* adjustment of K (upward in this case), rather than attempting an instantaneous maximization of $\tilde{\Pi}(K_t, L_t, I_t)$. In a continuous-time framework, to attempt the latter would in principle entail an instantaneous upward jump in K_t of size $\Delta K_t = a$ for some $a > 0$, requiring $I_t \cdot \Delta t = a$ for $\Delta t = 0$. This would require $I_t = \infty$, which implies $G(I_t, K_t) = \infty$. This implication may be interpreted either as such a jump being impossible or at least so costly that no firm would pursue it.

B. Necessary transversality conditions

Proof of necessity of (14.16) For convenience we name (14.16) (*) and repeat it here:

$$\lim_{t\to\infty} q_t e^{-\int_0^t (r_\tau + \delta) d\tau} = 0. \tag{*}$$

Consider an interior optimal path $(K_t, L_t, I_t)_{t=0}^\infty$, our reference path. According to the Maximum Principle, the path must for every $t \geq 0$ satisfy the first-order conditions (14.11), (14.12), and (14.13). Rearranging (14.13) and multiplying through by the factor $e^{-\int_0^t (r_\tau + \delta) d\tau}$, we get

$$[(r_t + \delta)q_t - \dot{q}_t]\, e^{-\int_0^t (r_\tau + \delta) d\tau} = (F_K(K_t, L_t) - G_K(I_t, K_t)) e^{-\int_0^t (r_\tau + \delta) d\tau}. \tag{14.53}$$

Integration on both sides of (14.53) yields

$$\int_0^T \left[(r_t+\delta)q_t - \dot{q}_t\right] e^{-\int_0^t (r_\tau+\delta)d\tau} = \int_0^T \left(F_K(K_t,L_t) - G_K(I_t,K_t)\right) e^{-\int_0^t (r_\tau+\delta)d\tau} dt. \tag{14.54}$$

In Lemma A1, let $f(t) = q_t$, $a(t) = r_t + \delta$, $t_0 = 0$, $t_1 = T$, and multiply through by -1 to get

$$\int_0^T \left[(r_t+\delta)q_t - \dot{q}_t\right] e^{-\int_0^t (r_\tau+\delta)d\tau} = q_0 - q_T e^{-\int_0^T (r_\tau+\delta)d\tau}.$$

Comparing with (14.53), we have

$$q_0 - q_T e^{-\int_0^T (r_\tau+\delta)d\tau} = \int_0^T \left(F_K(K_t,L_t) - G_K(I_t,K_t)\right) e^{-\int_0^t (r_\tau+\delta)d\tau} dt. \tag{14.55}$$

Rearranging and letting $T \to \infty$, gives

$$q_0 = \int_0^\infty \left(F_K(K_t,L_t) - G_K(I_t,K_t)\right) e^{-\int_0^t (r_\tau+\delta)d\tau} dt + \lim_{T\to\infty} q_T e^{-\int_0^T (r_\tau+\delta)d\tau}. \tag{14.56}$$

Now, suppose initial investment per time unit is reduced by one unit of account relative to the reference path over the short time interval $[0, \Delta t)$. Then the firm would save an amount approximately equal to $(1+G_I(I_0,K_0))\Delta t = q_0\Delta t$, in view of the first-order condition (14.12). If, contrary to (*), $\lim_{T\to\infty} q_T \exp(-\int_0^T (r_\tau + \delta)d\tau) > 0$, then the saved approximative amount, $q_0\Delta t$, would *exceed* the first term on the right-hand side of (14.56) multiplied by Δt, which represents the present value of the stream of forgone gains from this marginal unit of installed capital during the Δt time units. Since a *reduction* in investment would thus be beneficial to the firm, the firm would have *overinvested* in the original situation. If instead, contrary to (*), $\lim_{T\to\infty} q_T \exp(-\int_0^T (r_\tau + \delta)d\tau) < 0$, then the saved amount would *be less* than the present value of the stream of forgone gains from the marginal unit of installed capital during the Δt time units. Since an *increase* in investment would thus be beneficial to the firm, the firm would have *underinvested* in the original situation. This proves (*).

Together with (14.56), (*) implies

$$q_t = \int_t^\infty (F_K(K_s,L_s) - G_K(I_s,K_s)) e^{-\int_t^s (r_\tau+\delta)d\tau} ds. \tag{14.57}$$

This proves (14.17).[20]

[20] An equivalent approach to derivation of (*) and (14.17) can be based on the general solution formula for linear inhomogeneous first-order differential equations. Indeed, the first-order condition (14.13) provides such a differential equation in q_t.

Proof of necessity of (14.14) For convenience we name the transversality condition (14.14) (**) and repeat it here:

$$\lim_{t\to\infty} K_t q_t e^{-\int_0^t r_\tau d\tau} = 0. \tag{**}$$

In cases where along an optimal path, K_t is not bounded from above for $t \to \infty$, this transversality condition is stronger than (14.16) because the implied constraint on the long-run evolution of q_t is sharper.

To prove (**),[21] consider an interior optimal path $(K_t, L_t, I_t)_{t=0}^{\infty}$. As noted in Section 14.1.1, we may interpret the maximized V_0 as a function of the initial capital and initial time:

$$V^*(K_0, 0) = \int_0^\infty \left[F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t\right] e^{-\int_0^t r_\tau d\tau} dt \equiv \int_0^\infty \Phi(K_t, L_t, I_t, t) e^{-\int_0^t r_\tau d\tau} dt. \tag{14.58}$$

where, to save notation, we have introduced $\Phi(K_t, L_t, I_t, t) \equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t$.

By the "principle of optimality" all subtrajectories of an optimal trajectory must also be optimal. Thus, if we have found the fastest way to go from A to C and this path goes via B, the fastest way to go from B to C is to follow the B to C portion of our original A-B-C path. In the present case, the principle of optimality allows us to write, for arbitray $t \geq 0$,

$$V^*(K_0, 0) = \int_0^t \Phi(K_s, L_s, I_s, s) e^{-\int_0^s r_\tau d\tau} ds + e^{-\int_0^t r_\tau d\tau} V^*(K_t, t), \tag{14.59}$$

where $V^*(K_t, t)$ is defined in a way analogue to (14.58), but with (K_t, t) as starting point. We may rewrite (14.59) as

$$V^*(K_t, t) = e^{\int_0^t r_\tau d\tau} \left[V^*(K_0, 0) - \int_0^t \Phi(K_s, L_s, I_s, s) e^{-\int_0^s r_\tau d\tau} ds \right]. \tag{14.60}$$

This function, the *value function*, is thus well-defined for all $t \geq 0$, given the interior optimal path considered.

LEMMA B1 For all $t \geq 0$ and $K_t > 0$, the function $\Phi(K_t, L_t, I_t, t) \equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t$ has the properties: $\Phi_K \geq 0$, $\Phi(K_t, 0, 0, t) \geq 0$, and $\Phi(K_t, L_t, I_t, t)$ is concave in its first three arguments.
Proof. The stated properties of Φ follow by construction, given the properties of F and G stated in Section 14.1.1. $\square$

LEMMA B2 Consider an interior optimal path, $(K_t, L_t, I_t)_{t=0}^{\infty}$, with associated value function $V^*(K_t, t)$. Then:

[21] The following draws upon Weitzman (2003, Chapter 3).

(i) Along the path, the function Φ has the property $\Phi_K > 0$ for all $t \geq 0$.
(ii) $V^*(K_t, t)$ is a positive, increasing, and concave function of K_t for all $t \geq 0$.
(iii) $q_t > 0$ for all $t \geq 0$.
(iv) $\lim_{t\to\infty} V^*(K_t, t)e^{-\int_0^t r_\tau d\tau} = 0$.

Proof (sketch). (i) Along an interior optimal path, by definition, $K_t > 0$ and $L_t > 0$ for all $t \geq 0$. It follows, by definition of a neoclassical production function, that $F_K > 0$ along the path. Since $G_K \leq 0$, we then have $\Phi_K = F_K - G_K > 0$ for all $t \geq 0$. (ii) For all $t \geq 0$ and $K_t > 0$, a possible choice is $L_t = 0 = I_t$, in which case $\Phi = \Phi(K_t, 0, 0, t) \geq 0$ by Lemma B1. It follows that $V^*(K_t, t) \geq 0$ for all $t \geq 0$. We also have $\frac{1}{2}K_t > 0$ and so $V^*(\frac{1}{2}K_t, t) \geq 0$. Consequently, in view of $\Phi_K > 0$ from (i), $V^*(K_t, t) > V^*(\frac{1}{2}K_t, t) \geq 0$. This proves $V^*(K_t, t) > 0$ for all $t \geq 0$. As to the second part of (ii), note that an extra unit of capital has durability (although declining exponentially over time), that $\Phi_K > 0$, and that there is always the option of maintaining I and L unchanged. Hence, more installed capital is always better than less installed capital at the same time. This proves that $V^*(K_t, t)$ is increasing in K_t. The concavity of $V^*(K_t, t)$ with respect to K_t is more intricate, but proved in Weitzman (2003, pp. 79-81). (iii) In Section 14.1.1 we saw that at points where V^* is differentiable with respect to K_t, $\partial V^*(K_t, t)/\partial K_t$ equals the right-hand side of (14.57). Hence, $q_t = \partial V^*(K_t, t)/\partial K_t$. By (ii), $\partial V^*(K_t, t)/\partial K_t > 0$, and so $q_t > 0$. (iv) From (14.60) follows

$$\begin{aligned}
\lim_{t\to\infty} V^*(K_t, t)e^{-\int_0^t r_\tau d\tau} &= V^*(K_0, 0) - \lim_{t\to\infty} \int_0^t \Phi(K_s, L_s, I_s, s)e^{-\int_0^s r_\tau d\tau} ds \\
&= V^*(K_0, 0) - V^*(K_0, 0) = 0,
\end{aligned}$$

where the second equality is due to the considered path $(K_t, L_t, I_t)_{t=0}^{\infty}$ being an optimal path. $\square$

By concavity of $V^*(K_t, t)$ follows

$$\lim_{K\to 0} V^*(K, t) \leq V^*(K_t, t) + \frac{\partial V^*(K_t, t)}{\partial K_t}(0 - K_t), \tag{14.61}$$

because a concave function nowhere lies above its tangent. Substituting q_t into (14.61) and rearranging gives

$$q_t K_t \leq V^*(K_t, t) - \lim_{K\to 0} V^*(K, t).$$

Multiplying through by $e^{-\int_0^t r_\tau d\tau}$ and letting $t \to \infty$, we get

$$\lim_{t\to\infty} q_t K_t e^{-\int_0^t r_\tau d\tau} \leq \lim_{t\to\infty} V^*(K_t, t)e^{-\int_0^t r_\tau d\tau} - \lim_{K\to 0} V^*(K, t)e^{-\int_0^t r_\tau d\tau} = 0 - 0 = 0,$$

where we have used (iv) of Lemma B1 and that $r_\tau > 0$ at least in the long run. Since both q_t and K_t are nonnegative, so is the left-hand side of this weak inequality. This shows that (14.14) is necessary for optimality.

C. On different specifications of the adjustment costs

Sometimes in the literature installation costs, J, appear in a form different from the one focused on in this chapter. Abel and Blanchard (1983), followed by the textbooks Blanchard and Fischer (1989) and Barro and Sala-i-Martin (2004, pp. 152-160), introduce a function, ϕ, representing capital installation costs *per unit of investment* as a function of the investment-capital ratio. That is, total installation cost is $J = \phi(I/K)I$, where $\phi(0) = 0, \phi' > 0$. This implies that $J/K = \phi(I/K)(I/K)$. The right-hand side of this equation may be called $g(I/K)$, and then we are back at the formulation in Section 14.1. Indeed, defining $x \equiv I/K$, we have installation costs per unit of capital equal to $g(x) = \phi(x)x$, and assuming $\phi(0) = 0, \phi' > 0$, it holds that

$$\begin{aligned} g(x) &= 0 \text{ for } x = 0, \ g(x) > 0 \text{ for } x \neq 0, \\ g'(x) &= \phi(x) + x\phi'(x) \gtreqless 0 \text{ for } x \gtreqless 0, \ \text{respectively.} \end{aligned}$$

For the theory to work, we also need that $g''(x)$ $(= 2\phi'(x) + x\phi''(x)) > 0$. When $x \geq 0$, this inequality is guaranteed by the assumptions $\phi(0) = 0$ and $\phi' > 0$. But when $x < 0$, it is not guaranteed. Then the less graceful additional assumption $2\phi'(x) + x\phi''(x) > 0$ is needed.

Sometimes an alternative hypothesis is considered, namely that the capital installation cost $G(I, K)$ takes the form of a reduction in capital formation rather than in output. Then we may write

$$\dot{K} = I - G(I, K) - \delta K \equiv \Psi(I, K) - \delta K, \tag{14.62}$$

where the "capital installation function" $\Psi(I, K)$ is defined for $I \geq 0$ and has the properties $\Psi(0, K) = 0$, $0 < \Psi_I \leq 1 = \Psi_I(0, K)$, $\Psi_{II} < 0$, and $\Psi_K \geq 0$.[22] This approach is used in for instance Hayashi (1982) and the textbook Heijdra and Ploeg (2002). With $\Psi(I, K)$ homogeneous of degree one, we can define $\psi(I/K) \equiv \Psi(I/K, 1)$ and write capital accumulation as $\dot{K}/K = \psi(I/K) - \delta$, with $\psi' = \Psi_I$ and $\psi'' < 0$. In the next chapter we use this approach to describe housing construction. Apart from silence about situations with disinvestment, the approcah gives qualitatively similar results as the one we have used in this chapter.

[22] To be consistent with these properties, the G function should not be "too convex". For instance, our favorite example in this chapter, $G(I, K) = (\beta/2)I^2/K$, would *not* do. A certain class of CES functions will do for $\Psi(I, K)$, cf. Section 15.2.

Finally, some analysts, e.g., Abel (1990), assume that installation costs are a strictly convex function of *net* investment, $I-\delta K$. This agrees well with intuition if mere replacement investment occurs in a smooth way not involving new technology, work interruption, and reorganization. To the extent capital investment is lumpy because of indivisibilities and new technology, it seems more plausible to specify the installation costs as a convex function of *gross* investment.

D. Proof of Hayashi's theorem

The point of departure is the production-investment problem in Section 14.1.1. The value of the firm as seen from time t is

$$V_t = \int_t^{\infty}(F(K_s,L_s)-G(I_s,K_s)-w_sL_s-I_s)e^{-\int_t^s r_\tau d\tau}ds, \tag{14.63}$$

where the production function F is neoclassical and concave in $(K,\ L)$, and that the installation cost function G is convex in (I,K). We introduce the functions

$$\begin{aligned} A(K,L) &\equiv F(K,L)-(F_K(K,L)K+F_L(K,L)L), &(14.64)\\ B(I,K) &\equiv G_I(I,K)I+G_K(I,K)K-G(I,K). &(14.65)\end{aligned}$$

Then the cash-flow of the firm at time s can be written

$$\begin{aligned} R_s &= F(K_s,L_s)-F_L(K_s,L_s)L_s-G(I_s,K_s)-I_s\\ &= A(K_s,L_s)+F_K(K_s,L_s)K_s+B(I_s,K_s)-G_I(I_s,K_s)I_s-G_K(I_s,K_s)K_s-I_s,\end{aligned}$$

where we have used first $F_L(K_s,L_s)=w_s$ and next the definitions of $A(\cdot)$ and $B(\cdot)$ in (14.64) and (14.65), respectively. This allows us to decompose tha maximized value of the firm, the value function $V_t=V^*(K_t,t)$, the following way:

$$\begin{aligned} V^*(K_t,t) &= \int_t^{\infty}(A(K_s,L_s)+B(I_s,K_s))\,e^{-\int_t^s r_\tau d\tau}ds &(14.66)\\ &\quad+\int_t^{\infty}[(F_K(K_s,L_s)-G_K(I_s,K_s))K_s-(1+G_I(I_s,K_s))I_s]e^{-\int_t^s r_\tau d\tau}ds\\ &= \int_t^{\infty}(A(K_s,L_s)+B(I_s,K_s))e^{-\int_t^s r_\tau d\tau}ds+q_tK_t, &(14.67)\end{aligned}$$

where the last equality is implied by Lemma D1 below.

LEMMA D1 Consider the firm's problem in Section 14.1.1. The associated necessary transversality condition (14.14) implies that the term in the second line of (14.66) equals q_tK_t, when investment follows the optimal path.

Proof. We want to characterize a given optimal path $(K_s, I_s, L_s)_{s=t}^{\infty}$. Keeping t fixed and using s as our varying time variable, we have for all $s \geq 0$,

$$(F_K(K_s, L_s) - G_K(I_s, K_s))K_s - (1+G_I(I_s, K_s))I_s = [(r_s+\delta)q_s - \dot{q}_s]K_s - (1+G_I(I_s, K_s))I_s$$

$$= [(r_s + \delta)q_s - \dot{q}_s]K_s - q_s(\dot{K}_s + \delta K_s) = r_s q_s K_s - (\dot{q}_s K_s + q_s \dot{K}_s) \equiv r_s u_s - \dot{u}_s,$$

where we have used (14.13), (14.12), (14.8), and the definition $u_s \equiv q_s K_s$. Defining $\xi_s \equiv (F_K(K_s, L_s) - G_K(I_s, K_s))K_s - (1 + G_I(I_s, K_s))I_s$, the function u_s thus satisfies the differential equation: $\dot{u}_s - r_s u_s = -\xi_s$. The solution to this linear differential equation is

$$u_s = \left(u_t - \int_t^s \xi_z e^{-\int_t^z r_\tau d\tau} dz\right) e^{\int_t^s r_\tau d\tau}.$$

By multiplying through by $e^{-\int_t^s r_\tau d\tau}$, rearranging, and inserting the definitions of u and ξ, we get

$$\begin{aligned} & \int_t^s [(F_K(K_z, L_z) - G_K(I_z, K_z))K_z - (1 + G_I(I_z, K_z))I_z] e^{-\int_t^z r_\tau d\tau} dz \\ = \; & q_t K_t - q_s K_s e^{-\int_t^s r_\tau d\tau} \rightarrow q_t K_t \quad \text{for} \quad s \rightarrow \infty, \end{aligned}$$

by the transversality condition (14.14) with t replaced by s and 0 replaced by t. □

For convenience, from Section 14.2 we repeat:

THEOREM (Hayashi) Assume that the firm is a price taker, that the production function F is neoclassical and concave in (K, L), and that the installation cost function G is convex in (I, K). Then, along the optimal path we have:

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in (K, L) and/or G is strictly convex in (I, K).

Proof. Isolating q_t in (14.67), it follows that

$$q_t^m \equiv q_t = \frac{V^*(K_t, t)}{K_t} - \frac{1}{K_t} \int_t^\infty [A(K_s, L_s) + B(I_s, K_s)] e^{-\int_t^s r_\tau d\tau} ds, \qquad (14.68)$$

when moving along the optimal path. Since F is a concave $\mathbb{C}^1$ function and, as a production function, has $F(0, 0) = 0$, we have for all K and L, $A(K, L) \geq 0$ with equality sign, if and only if F is homogeneous of degree one. Similarly, since G is a convex $\mathbb{C}^1$ function and has $G(0, 0) = 0$, we have for all I and K, $B(I, K) \geq 0$ with equality sign, if and only if G is homogeneous of degree one. Now the conclusions (i) and (ii) follow from (14.68) and the definition $q_t^a \equiv V^*(K_t,t)/K_t$ from (14.32). $\square$

Concerning item (i) of Hayashi's theorem, a simple – and perhaps more illuminating – way to understand it is the following (based on D'Autume and Michel, 1985). Suppose G and F are homogeneous of degree one. Then $A = B = 0$, $G(I,K) = g(I/K)K$, and $F(K,L) = F_K(K,L)K + F_L(K,L)L = f'(k)K + wL$, hence

$$F(K,L) - wL = f'(k)K, \quad \text{and} \quad F_L(K,L) = f(k) - kf'(k) = w, \tag{14.69}$$

where f is the production function in intensive form and where the first-order condition $F_L(K,L) = w$ has been applied. Consider an optimal path $(K_s, I_s, L_s)_{s=t}^{\infty}$ and let $k_s \equiv K_s/L_s$ and $x_s \equiv I_s/K_s$ along this path which we now want to characterize. As the path is assumed optimal, from (14.63) and (14.69) follows

$$V_t = V^*(K_t,t) = \int_t^\infty [f'(k_s) - g(x_s) - x_s]K_s e^{-\int_t^s r_\tau d\tau} ds. \tag{14.70}$$

From $\dot{K}_t = (x_t - \delta)K_t$ follows $K_s = K_t \exp(\int_t^s (x_\tau - \delta)d\tau)$. Substituting this into (14.70) yields

$$V^*(K_t,t) = K_t \int_t^\infty [f'(k_s) - g(x_s) - x_s]e^{-\int_t^s (r_\tau - x_\tau + \delta)d\tau} ds.$$

In view of (14.28), the optimal investment *ratio* x_s depends, for all s, only on q_s, not on K_s and so not on K_t. Similarly, in view of (14.69), for all s the chosen k_s depends only on the market wage w_s, not on K_s and so not on K_t. Hence,

$$\frac{\partial V^*(K_t,t)}{\partial K_t} = \int_t^\infty [f'(k_s) - g(x_s) - x_s]e^{-\int_t^s (r_\tau - x_\tau + \delta)d\tau} ds = \frac{V^*(K_t,t)}{K_t}.$$

From the definitions (14.33) and (14.32), we now conclude $q_t^m = q_t^a$.

Remark. We have assumed throughout that G is strictly convex in I. This does not imply that G is strictly convex in (I,K). For example, the function $G(I,K) = I^2/K$ is strictly convex in I (since $G_{II} = 2/K > 0$). But at the same time this function has $B(I,K) = 0$ and is therefore homogeneous of degree one. Hence, it is not strictly convex in (I,K).

E. The slope of the $\dot{q} = 0$ locus in the SOE case

For Case 1 we shall determine the sign of the slope of the $\dot{q} = 0$ locus in the case $g + n = 0$, considered in Fig. 14.4. Substitute $\varphi(q) \equiv -[g(m(q)) - (q-1)m(q)]$

into (14.42) and on both sides take the total differential with respect to K and q to get

$$0 = -F_{KK}(K, \bar{L})dK + [r + \delta - m(q)]\,dq,$$

where we have applied that $\varphi'(q) = m(q)$, see Lemma 2 of Section 14.1.3. Hence

$$\frac{dq}{dK}_{|\dot{q}=0} = \frac{F_{KK}(K, \bar{L})}{r + \delta - m(q)} \quad \text{for } m(q) \neq r + \delta.$$

From this it is not possible to sign dq/dK at all points along the $\dot{q} = 0$ locus. But in a small neighborhood of the steady state we have $m(q) \approx \delta$, hence $r+\delta-m(q) \approx r > 0$. And since $F_{KK} < 0$, this implies that at least in a small neighborhood of E in Fig. 14.4, the $\dot{q} = 0$ locus is negatively sloped.

In Case 2, consider the case $g + n > 0$, illustrated in Fig. 14.6. Here we get in a similar way

$$\frac{dq}{d\tilde{k}}_{|\dot{q}=0} = \frac{f''(\tilde{k}^*)}{r + \delta - m(q)} \quad \text{for } m(q) \neq r + \delta.$$

From this it is not possible to sign $dq/d\tilde{k}$ at all points along the $\dot{q} = 0$ locus. But in a small neighborhood of the steady state, we have $m(q) \approx \delta + \gamma + n$, hence $r + \delta - m(q) \approx r - \gamma - n > 0$, where the parameter inequality was assumed in the text. Since $f'' < 0$, then, at least in a small neighborhood of E in Fig. 14.6, the $\dot{q} = 0$ locus is negatively sloped, when $r > \gamma + n$.

F. The divergent paths (Section 14.3.2)

It is enough to consider Case 1, as Case 2 is in this respect similar. From the differential equations (14.39) and (14.40) follows that $d(q_t K_t)/dt =$

$$\begin{aligned} q_t\dot{K}_t + \dot{q}_t K_t &= \left[q_t(m(q_t) - \delta) + (r + \delta)q_t - F_K(K_t, \bar{L}) + g(m(q_t)) - (q_t - 1)m(q_t)\right] K_t \\ &= \left[rq_t - F_K(K_t, \bar{L}) + m(q_t) + g(m(q_t))\right] K_t \\ &= \left[r + (m(q_t) - F_K(K_t, \bar{L}) + g(m(q_t)))/q_t\right] K_t q_t. \qquad (*) \end{aligned}$$

There are two categories of divergent paths, those that ultimately enter Region II in Fig. 14.4 and move north-east and those that ultimately enter Region IV and move south-west. Consider the first category. After entering Region II, we have $\dot{K}_t > 0$ and $\dot{q}_t > 0$. Since $\dot{K}_t/K_t = m(q_t) - \delta$ is positive as well as growing, K_t has no upper bound. So, by the upper Inada condition, $F_K \to 0$ for $K \to \infty$, the positive term in square brackets in (*) will sooner or later necessarily exceed r, thus implying that $q_t K_t$ grows at a rate higher than r. This violates the transversality condition (14.41) and is therefore not consistent with intertemporal profit maximization, hence not with an equilibrium path under perfect foresight.

Paths starting *below* the saddle path will *not* violate the transversality condition. Indeed, in view of free disposal, an optimal q can never be negative. These paths will, however, for all t have q_t below q^* and therefore, as detailed in the text, not fully exploit the potential benefits of capital.

Nevertheless, let us see how in detail paths ultimately entering Region IV behave. After entering Region IV, say at $t = t_0 \geq 0$, such a path has $\dot{K}_t < 0$ and $\dot{q}_t < 0$. Differentiating with respect to t on both sides of (14.40), after having substituted $\varphi(q) \equiv -[g(m(q)) - (q-1)m(q)]$, gives

$$\begin{aligned}
\ddot{q}_t &= (r+\delta)\dot{q}_t - F_{KK}(K_t, \bar{L})(m(q_t) - \delta)K_t - \varphi'(q_t)\dot{q}_t \\
&= (r+\delta)\dot{q}_t - F_{KK}(K_t, \bar{L})(m(q_t) - \delta)K_t - m(q_t)\dot{q}_t \text{ (from Lemma 2 in Section 14.1)} \\
&< (r+\delta - m(q_t))\dot{q}_t \qquad \text{(since } F_{KK} < 0 \text{ and } m(q_t) < \delta \text{ for } q_t < q^*) \\
&< 0 \text{ for all } t,
\end{aligned}$$

the latter inequality coming from $\dot{q}_t < 0$ and $r+\delta-m(q_t) > r+\delta-m(q^*) = r > 0$ in view of $q_t < q^*$. So, in Region IV, we have, for all t, that not only is $\dot{q}_t < 0$, but $\dot{q}_t$ remains bounded away from 0. Hence, there exists a $t_1 > t_0 \geq 0$ such that

$$q_{t_1} = q_{t_0} + \int_{t_0}^{t_1} \dot{q}_t dt = 0.$$

Also K is decreasing in Region IV. Might K_t reach zero before q_t does? No. We prove this by contradiction. Suppose that the smallest t for which $K_t = 0$ is $t = \tau < t_1$. Then, for all $t \in (t_0, \tau]$, we have

$$0 > \dot{K}_t = I_t - \delta K_t = (m(q_t) - \delta)K_t > (m(q(\tau)) - \delta)K_t, \tag{14.71}$$

where the second inequality is due to $\dot{q}_t < 0$ and $m'(q) > 0$. The solution to the differential equation for K_t in (14.71) thus satisfies the inequality

$$K_t > K_{t_0} e^{(m(q(\tau))-\delta)(t-t_0)} > 0 \text{ for all } t \in (t_0, \tau].$$

This contradicts that $K_\tau = 0$, and we conclude that $K_t > 0$ for all $t < t_1$.

We know from Technical remark at the end of Section 14.1.2, that if the optimal q really is 0, then the firm can do no better at time t_1 than letting $I_{t_1} = 0$. And since an optimal q cannot be negative, we have for $t \geq t_1$, $q_t = 0$ and $\dot{K}_t = -\delta K_t$, so that $K_t \to 0$ for $t \to \infty$.

14.7 Exercises

14.1

14.2 Let F be Cobb-Douglas with CRS and let $G(I) = (\beta/2)I^2$, $\beta > 0$. a) Find q^*, I, L, and K along the optimal path. *Hint:* the differential equation $\dot{x}(t) + ax(t) = b$ with $a \neq 0$ has the solution $x(t) = (x(0) - x^*)e^{-at} + x^*$, where $x^* = b/a$. b) Evaluate the model.

14.3 (see end of Section 14.3)