Chapter 14

Fixed capital investment and Tobin's q

The models considered so far (the OLG models as well as the representative agent models) have ignored capital adjustment costs. In the closed-economy version of the models aggregate investment is merely a reflection of aggregate saving and appears in a "passive" way as just the residual of national income after households have chosen their consumption. We can describe what is going on by telling a story in which firms just rent capital goods owned by the households and households save by purchasing additional capital goods. In these models only households solve intertemporal decision problems. Firms merely demand labor and capital services with a view to maximizing current profits. This may be a legitimate abstraction in some contexts within long-run analysis. In short- and medium-run analysis, however, the dynamics of fixed capital investment is important. So a more realistic approach is desirable.

In the real world the capital goods used by a production firm are usually owned by the firm itself rather than rented for single periods on rental markets. This is because inside the specific plant in which these capital goods are an integrated part, they are generally worth much more than outside. So in practice firms acquire and install fixed capital equipment to maximize discounted expected earnings in the future.

Tobin's *q-theory of investment* (after the American Nobel laureate James Tobin, 1918-2002) is an attempt to model these features. In this theory,

(a) *firms* make the *investment decisions* and *install* the purchased capital goods in their own businesses;

(b) there are certain *adjustment costs* associated with this investment: in addition to the direct cost of buying new capital goods there are costs of installation, costs of reorganizing the plant, costs of retraining workers to operate the new machines etc.;

(c) the adjustment costs are *strictly convex* so that marginal adjustment costs are increasing in the level of investment − think of constructing a plant in a month rather than a year.

The strict convexity of adjustment costs is the crucial constituent of the theory. It is that element which assigns investment decisions an *active* role in the model. There will be both a well-defined saving decision and a well-defined investment decision, separate from each other. Households decide the saving, firms the physical capital investment; households accumulate financial assets, firms accumulate physical capital. As a result, in a closed economy interest rates have to adjust for aggregate demand for goods (consumption plus investment) to match aggregate supply of goods. The role of interest rate changes is no longer to clear a rental market for capital goods.

To fix the terminology, from now the adjustment costs of setting up new capital equipment in the firm and the associated costs of reorganizing work processes will be subsumed under the term *capital installation costs.* When faced with strictly convex installation costs, the optimizing firm has to take the *future* into account, that is, firms' forward-looking *expectations* become important. To smooth out the adjustment costs, the firm will adjust its capital stock only *gradually* when new information arises. We thereby avoid the counterfactual implication from earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world financial market changes. Moreover, sluggishness in investment is exactly what the data show. Some empirical studies conclude that only a third of the difference between the current and the "desired" capital stock tends to be covered within a year (Clark 1979).

The q-theory of investment constitutes one approach to the explanation of this sluggishness in investment. Under certain conditions, to be described below, the theory gives a remarkably simple operational macroeconomic investment function, in which the key variable explaining aggregate investment is the valuation of the firms by the stock market relative to the replacement value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

14.1 Convex capital installation costs

Let the technology of a single firm be given by

$$\tilde{Y} = F(K, L),$$

where $\tilde{Y}, K$, and L are "potential output" (to be explained), capital input, and labor input per time unit, respectively, while F is a concave neoclassical production function. So we allow decreasing as well as constant returns to scale (or a combination of locally CRS and locally DRS), whereas increasing returns to scale is ruled out. Until further notice technological change is ignored for simplicity. Time is continuous. The dating of the variables will not be explicit unless needed for clarity. The increase per time unit in the firm's capital stock is given by

$$\dot{K} = I - \delta K, \qquad \delta > 0, \tag{14.1}$$

where I is gross fixed capital investment per time unit and δ is the rate of wearing down of capital (physical capital depreciation). To fix ideas, we presume the realistic case with positive capital depreciation, but most of the results go through even for $\delta = 0$.

Let J denote the firm's capital installation costs (measured in units of output) per time unit. The installation costs imply that a part of the potential output, $\tilde{Y}$, is "used up" in transforming investment goods into installed capital; only $\tilde{Y} - J$ is "true output" available for sale.

Assuming the price of investment goods is one (the same as that of output goods), then total investment costs per time unit are $I + J$, i.e., the direct purchase costs, $1 \cdot I$, plus the indirect cost associated with installation etc., J. The q-theory of investment assumes that the capital installation cost, J, is a strictly convex function of gross investment and is either independent of or a decreasing function of the current capital stock. Thus,

$$J = G(I, K),$$

where the installation cost function G satisfies

$$G(0, K) = 0, \; G_I(0, K) = 0, \; G_{II}(I, K) > 0, \text{ and } G_K(I, K) \leq 0 \tag{14.2}$$

for all K and all (I, K), respectively. For fixed $K = \bar{K}$ the graph is as shown in Fig. 14.1. Also negative gross investment, i.e., sell off of capital equipment, involves costs (for dismantling, reorganization etc.). Therefore $G_I < 0$ for $I < 0$. The important assumption is that $G_{II} > 0$ (strict convexity in I), implying that the marginal installation cost is increasing in the level of gross investment. If the firm wants to accomplish a given installation project in only half the time, then the installation costs are more than doubled (the risk of mistakes is larger, the problems with reorganizing work routines are larger etc.).

The strictly convex graph in Fig. 14.1 illustrates the essence of the matter. Assume the current capital stock in the firm is $\bar{K}$ and that the firm wants to increase it by a given amount $\overline{\Delta K}$. If the firm chooses the investment level $\bar{I} >$

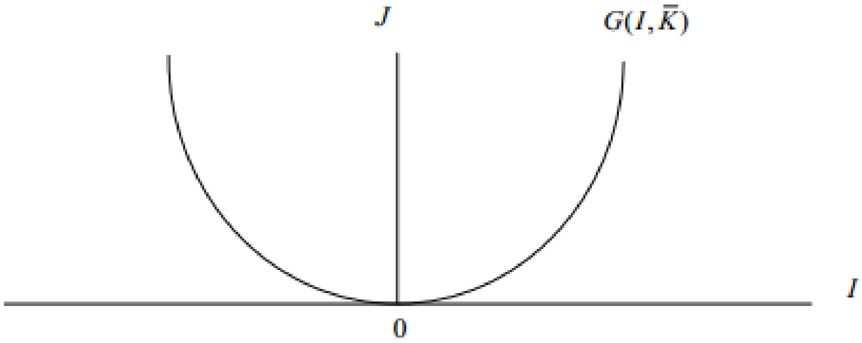

Figure 14.1: Installation costs as a function of gross investment when $K = \bar{K}$.

0 per time unit in the time interval $[t, t + \Delta t)$, then, in view of (14.1), $\Delta K \approx (\bar{I} - \delta\bar{K})\Delta t$. So it takes $\Delta t \approx \overline{\Delta K}/(\bar{I} - \delta\bar{K})$ units of time to accomplish the desired increase $\overline{\Delta K}$. If, however, the firm slows down the adjustment and invests only half of $\bar{I}$ per time unit, then it takes approximately twice as long time to accomplish $\overline{\Delta K}$. Total costs of the two alternative courses of action are approximately $G(\bar{I}, \bar{K})\Delta t$ and $G(\frac{1}{2}\bar{I}, \bar{K})2\Delta t$, respectively (ignoring discounting and assuming the initial increase in capital is small in relation to $\bar{K}$). By drawing a few straight line segments in Fig. 14.1 the reader will be convinced that the last-mentioned cost is smaller than the first-mentioned due to strict convexity of installation costs (see Exercise 14.1). *Haste is waste.*

On the other hand, there are of course limits to how slow the adjustment to the desired capital stock should be. Slower adjustment means postponement of the potential benefits of a higher capital stock. So the firm faces a trade-off between fast adjustment to the desired capital stock and low adjustment costs.

In addition to the strict convexity of G with respect to I, (14.2) imposes the condition $G_K(I, K) \leq 0$. Indeed, it often seems realistic to assume that $G_K(I, K) < 0$ for $I \neq 0$. A given amount of investment may require more reorganization in a small firm than in a large firm (size here being measured by K). When installing a new machine, a small firm has to stop production altogether, whereas a large firm can to some extent continue its production by shifting some workers to another production line. A further argument is that the more a firm has invested historically, the more experienced it is now. So, for a given I today, the associated installation costs are lower, given a larger accumulated K.

14.1.1 The decision problem of the firm

In the absence of tax distortions, asymmetric information, and problems with enforceability of financial contracts, the Modigliani-Miller theorem (Modigliani and Miller, 1958) says that the financial structure of the firm is both indeterminate and irrelevant for production decisions (see Appendix A). Although the conditions required for this theorem are very idealized, the q-theory of investment accepts them because they allow the analyst to concentrate on the production aspects in a first approach.

With the output good as unit of account, let the operating cash flow (the net payment stream to the firm before interest payments on debt, if any) at time t be denoted R_t (for "receipts"). Then

$$R_t \equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \tag{14.3}$$

where w_t is the wage per unit of labor at time t. As mentioned, the installation cost $G(I_t, K_t)$ implies that a part of production, $F(K_t, L_t)$, is used up in transforming investment goods into installed capital; only the difference $F(K_t, L_t) - G(I_t, K_t)$ is available for sale.

We ignore uncertainty and assume the firm is a price taker. The interest rate is r_t, which we assume to be positive, at least in the long run. The decision problem, as seen from time 0, is to choose a plan $(L_t, I_t)_{t=0}^{\infty}$ so as to maximize the firm's *market value*, i.e., the present value of the future stream of expected cash flows:

$$\max_{(L_t, I_t)_{t=0}^{\infty}} V_0 = \int_0^{\infty} R_t e^{-\int_0^t r_s ds} dt \quad \text{s.t. (14.3) and} \tag{14.4}$$

$$L_t \geq 0, I_t \text{ free (i.e., no restriction on } I_t), \tag{14.5}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{14.6}$$

$$K_t \geq 0 \text{ for all } t. \tag{14.7}$$

There is no specific terminal condition but we have posited the feasibility condition (14.7) saying that the firm can never have a negative capital stock.[1]

In the previous chapters the firm was described as solving a series of static profit maximization problems. Such a description is no longer valid, however, when there is dependence across time, as is the case here. When installation

[1]It is assumed that w_t is a piecewise continuous function. At points of discontinuity (if any) in investment, we will consider investment to be a *right-continuous* function of time. That is, $I_{t_0} = \lim_{t \to t_0^+} I_t$. Likewise, at such points of discontinuity, by the "time derivative" of the corresponding state variable, K, we mean the *right-hand* time derivative, i.e., $\dot{K}_{t_0} = \lim_{t \to t_0^+} (K_t - K_{t_0})/(t - t_0)$. Mathematically, these conventions are inconsequential, but they help the intuition.

costs are present, current decisions depend on the expected future circumstances. The firm makes a plan for the whole future so as to maximize the value of the firm, which is what matters for the owners. This is the general neoclassical hypothesis about firms' behavior. As shown in Appendix A, when strictly convex installation costs or similar dependencies across time are absent, then value maximization is equivalent to solving a sequence of static profit maximization problems, and we are back in the previous chapters' description.

To solve the problem (14.4) − (14.7), where R_t is given by (14.3), we apply the Maximum Principle. The problem has two control variables, L and I, and one state variable, K. We set up the current-value Hamiltonian:

$$H(K, L, I, q, t) \equiv F(K, L) - wL - I - G(I, K) + q(I - \delta K), \tag{14.8}$$

where q (to be interpreted economically below) is the adjoint variable associated with the dynamic constraint (14.6). For each $t \geq 0$ we maximize H w.r.t. the control variables. Thus, $\partial H/\partial L = F_L(K, L) - w = 0$, i.e.,

$$F_L(K, L) = w; \tag{14.9}$$

and $\partial H/\partial I = -1 - G_I(I, K) + q = 0$, i.e.,

$$1 + G_I(I, K) = q. \tag{14.10}$$

Next, we partially differentiate H w.r.t. the state variable and set the result equal to $rq - \dot{q}$, where r is the discount rate in (14.4):

$$\frac{\partial H}{\partial K} = F_K(K, L) - G_K(I, K) - q\delta = rq - \dot{q}. \tag{14.11}$$

Then, the Maximum Principle says that for an interior optimal path (K_t, L_t, I_t) there exists an adjoint variable q, which is a continuous function of t, written q_t, such that for all $t \geq 0$ the conditions (14.9), (14.10), and (14.11) hold and the transversality condition

$$\lim_{t\to\infty} K_t q_t e^{-\int_0^t r_s ds} = 0 \tag{14.12}$$

is satisfied.

The optimality condition (14.9) is the usual employment condition equalizing the marginal product of labor to the real wage. In the present context with strictly convex capital installation costs, this condition attains a distinct role as labor will in the short run be the only variable input. This is because the strictly convex capital installation costs imply that the firm's installed capital in the short run is a quasi-fixed production factor. So, effectively there are diminishing returns (equivalent with rising marginal costs) in the short run even though the production function might have CRS.

The left-hand side of (14.10) gives the cost of acquiring one extra unit of installed capital at time t (the sum of the cost of buying the marginal investment good and the cost of its installation). That is, the left-hand side is the marginal cost, MC, of increasing the capital stock in the firm. Since (14.10) is a necessary condition for optimality, the right-hand side of (14.10) must be the marginal benefit, MB, of increasing the capital stock. Hence, q_t represents the value to the optimizing firm of having one more unit of (installed) capital at time t. To put it differently: the adjoint variable q_t can be interpreted as the shadow price (measured in current output units) of capital along the optimal path.[2]

As to the interpretation of the differential equation (14.11), a condition for optimality must be that the firm acquires capital up to the point where the "marginal productivity of capital", $F_K - G_K$, equals "capital costs", $r_t q_t + (\delta q_t - \dot{q}_t)$; the first term in this expression represents interest costs and the second economic depreciation. In (14.11) the "marginal productivity of capital" appears as $F_K - G_K$, because we should take into account the potential reduction, $-G_K$, of installation costs in the next instant brought about by the marginal unit of already installed capital. The shadow price q_t appears as the "overall" price at which the firm can buy and sell the marginal unit of installed capital. In fact, in view of $q_t = 1 + G_I(K_t, L_t)$ along the optimal path (from (14.10)), q_t *measures*, approximately, *both* the "overall" cost increase associated with *increasing* investment by one unit and the "overall" cost saving associated with *decreasing* investment by one unit. In the first case the firm not only has to pay one extra unit of account in the investment goods market but must also bear an installation cost equal to $G_I(K_t, L_t)$, thereby in total investing q_t units of account. And in the second case the firm recovers q_t by saving both on installation costs and purchases in the investment goods market. Continuing along this line of thought, by reordering in (14.11) we get the "no-arbitrage" condition

$$\frac{F_K - G_K - \delta q + \dot{q}}{q} = r, \tag{14.13}$$

saying that along the optimal path the rate of return on the marginal unit of installed capital must equal the interest rate.

The transversality condition (14.12) says that the present value of the capital stock "left over" at infinity must be zero. That is, the capital stock should not in the long run grow too fast, given the evolution of its discounted shadow price. In addition to necessity of (14.12) it can be shown[3] that the discounted shadow

[2] Recall that a *shadow price,* measured in some unit of account, of a good, from the point of view of the buyer, is the maximum number of units of account that he or she is willing to offer for one extra unit of the good.

[3] See Appendix B.

price itself in the far future must along an optimal path be asymptotically nil, i.e.,

$$\lim_{t\to\infty} q_t e^{-\int_0^t r_s ds} = 0. \tag{14.14}$$

If along the optimal path, K_t grows without bound, then not only must (14.14) hold but, in view of (14.12), the discounted shadow price must in the long run approach zero *faster* than K_t grows. Intuitively, otherwise the firm would be "over-accumulating". The firm would gain by reducing the capital stock "left over" for eternity (which is like"money left on the table"), since reducing the ultimate investment and installation costs would raise the present value of the firm's expected cash flow.

In connection with (14.10) we claimed that q_t can be interpreted as the shadow price (measured in current output units) of capital along the optimal path. A confirmation of this interpretation is obtained by solving the differential equation (14.11). Indeed, multiplying by $e^{-\int_0^t (r_s+\delta)ds}$ on both sides of (14.11), we get by integration and application of (14.14),[4]

$$q_t = \int_t^{\infty} \left[F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)\right] e^{-\int_t^\tau (r_s+\delta)ds} d\tau. \tag{14.15}$$

The right-hand side of (14.15) is the present value, as seen from time t, of expected future increases of the firm's cash-flow that would result if one extra unit of capital were installed at time t; indeed, $F_K(K_\tau, L_\tau)$ is the direct contribution to output of one extra unit of capital, while $-G_K(I_\tau, K_\tau) \geq 0$ represents the potential reduction of installation costs in the next instant brought about by the marginal unit of installed capital. However, future increases of cash-flow should be discounted at a rate equal to the interest rate *plus* the capital depreciation rate; from one extra unit of capital at time t there are only $e^{-\delta(\tau-t)}$ units left at time τ.

To concretize our interpretation of q_t as representing the value to the optimizing firm at time t of having one extra unit of installed capital, let us make a thought experiment. Assume that a extra units of installed capital at time t drops down from the sky. At time $\tau > t$ there are $a \cdot e^{-\delta(\tau-t)}$ units of these still in operation so that the stock of installed capital is

$$K'_\tau = K_\tau + a \cdot e^{-\delta(\tau-t)}, \tag{14.16}$$

where K_τ denotes the stock of installed capital as it would have been without this "injection". Now, in (14.3) replace t by τ and consider the optimizing firm's

[4]For details, see Appendix A.

cash-flow R_τ as a function of $(K_\tau, L_\tau, I_\tau, \tau, t, a)$. Taking the partial derivative of R_τ w.r.t. a at the point $(K_\tau, L_\tau, I_\tau, \tau, t, 0)$, we get

$$\frac{\partial R_\tau}{\partial a}_{|a=0} = \left[F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)\right] e^{-\delta(\tau - t)}. \tag{14.17}$$

Considering the value of the optimizing firm at time t as a function of installed capital, K_t, and t itself, we denote this function $V^*(K_t, t)$. Then at any point where V^* is differentiable, we have

$$\begin{aligned} \frac{\partial V^*(K_t, t)}{\partial K_t} &= \int_t^\infty \left(\frac{\partial R_\tau}{\partial a}_{|a=0}\right) e^{-\int_t^\tau r_s ds} d\tau \\ &= \int_t^\infty [F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)] e^{-\int_t^\tau (r_s+\delta) ds} d\tau = q_t \end{aligned} \tag{14.18}$$

when the firm moves along the optimal path. The second equality sign comes from (14.17) and the third is implied by (14.15). So the value of the adjoint variable, q, at time t equals the contribution to the firm's maximized value of a fictional marginal "injection" of installed capital at time t. This is just another way of saying that q_t represents the benefit to the firm of the marginal unit of installed capital along the optimal path.

This story facilitates the understanding that the control variables at any point in time should be chosen so that the Hamiltonian function is maximized. Thereby one maximizes the properly weighted sum of the current direct contribution to the criterion function and the indirect contribution, which is the benefit (as measured approximately by $q_t \Delta K_t$) of having a higher capital stock in the future.

As we know, the Maximum Principle gives only necessary conditions for an optimal path, not sufficient conditions. We use the principle as a tool for finding candidates for a solution. Having found in this way a candidate, one way to proceed is to check whether Mangasarian's sufficient conditions are satisfied. Given the transversality condition (14.12) and the non-negativity of the state variable, K, the only additional condition to check is whether the Hamiltonian function is jointly concave in the endogenous variables (here K, L, and I). If it is jointly concave in these variables, then the candidate *is* an optimal solution. Owing to concavity of $F(K, L)$, inspection of (14.8) reveals that the Hamiltonian function is jointly concave in (K, L, I) if $-G(I, K)$ is jointly concave in (I, K). This condition is equivalent to $G(I, K)$ being jointly convex in (I, K), an assumption allowed within the confines of (14.2); for example, $G(I, K) = (\frac{1}{2})\beta I^2/K$ as well as the simpler $G(I, K) = (\frac{1}{2})\beta I^2$ (where in both cases $\beta > 0$) will do. Thus, assuming joint convexity of $G(I, K)$, the first-order conditions and the transversality condition are not only necessary, but also sufficient for an optimal solution.

14.1.2 The implied investment function

From condition (14.10) we can derive an investment function. Rewriting (14.10), we have that an optimal path satisfies

$$G_I(I_t, K_t) = q_t - 1. \tag{14.19}$$

Combining this with the assumption (14.2) on the installation cost function, we see that

$$I_t \gtreqless 0 \text{ for } q_t \gtreqless 1, \text{ respectively,} \tag{14.20}$$

cf. Fig. 14.2.[5] In view of $G_{II} \neq 0$, (14.19) implicitly defines optimal investment, I_t, as a function of the shadow price, q_t, and the state variable, K_t :

$$I_t = \mathcal{M}(q_t, K_t), \tag{14.21}$$

where, in view of (14.20), $\mathrm{M}(1, K_t) = 0$. By implicit differentiation w.r.t. q_t and K_t, respectively, in (14.19), we find

$$\frac{\partial I_t}{\partial q_t} = \frac{1}{G_{II}(I_t, K_t)} > 0, \qquad \text{and} \qquad \frac{\partial I_t}{\partial K_t} = -\frac{G_{IK}(I_t, K_t)}{G_{II}(I_t, K_t)},$$

where the latter cannot be signed without further specification.

It follows that optimal investment is an increasing function of the shadow price of installed capital. In view of (14.20), $\mathcal{M}(1, K) = 0$. Not surprisingly, the investment rule is: invest now, if and only if the value to the firm of the marginal unit of installed capital is larger than the price of the capital good (which is 1, excluding installation costs). At the same time, the rule says that, because of the convex installation costs, invest only up to the point where the marginal installation cost, $G_I(I_t, K_t)$, equals $q_t - 1$, cf. (14.19).

Condition (14.21) shows the remarkable information content that the shadow price q_t has. As soon as q_t is known (along with the current capital stock K_t), the firm can decide the optimal level of investment through knowledge of the installation cost function G alone (since, when G is known, so is in principle the inverse of G_I w.r.t. I, the investment function $\mathcal{M}$). All the information about the production function, input prices, and interest rates now and in the future that is relevant to the investment decision is summarized in one number, q_t. The form of the investment function, $\mathcal{M}$, depends only on the installation cost function G. These are very useful properties in theoretical and empirical analysis.

[5] From the assumptions made in (14.2), we only know that the graph of $G_I(I, \bar{K})$ is an upward-sloping curve going through the origin. Fig. 14.2 shows the special case where this curve happens to be linear.

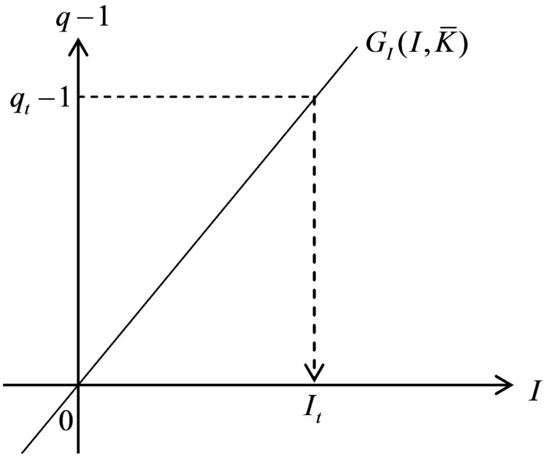

Figure 14.2: Marginal installation costs as a function of the gross investment level, I, for a given amount, $\bar{K}$, of installed capital. The optimal gross investment, I_t, when $q = q_t$ is indicated.

14.1.3 A not implausible special case

We now introduce the convenient case where the installation function G is homogeneous of degree one w.r.t. I and K so that we can, for $K > 0$, write

$$\begin{aligned} J &= G(I, K) = G(\frac{I}{K}, 1)K \equiv g(\frac{I}{K})K, \qquad \text{or} \qquad (14.22) \\ \frac{J}{K} &= g(\frac{I}{K}), \end{aligned}$$

where $g(\cdot)$ represents the installation cost-capital ratio and $g(0) \equiv G(0, 1) = 0$, by (14.2).

LEMMA 1 The function $g(\cdot)$ has the following properties:
(i) $g'(I/K) = G_I(I, K)$;
(ii) $g''(I/K) = G_{II}(I, K)K > 0$ for $K > 0$; and
(iii) $g(I/K) - g'(I/K)I/K = G_K(I, K) < 0$ for $I \neq 0$.

Proof. (i) $G_I = Kg'/K = g'$; (ii) $G_{II} = g''/K$; (iii) $G_K = \partial(g(I/K)K)/\partial K = g(I/K) - g'(I/K)I/K < 0$ for $I \neq 0$ since, in view of $g'' > 0$ and $g(0) = 0$, we have $g(x)/x < g'(x)$ for all $x \neq 0$. □

The graph of $g(I/K)$ is qualitatively the same as that in Fig. 14.1 (imagine we have $\bar{K} = 1$ in that graph). The installation cost relative to the existing capital stock is now a strictly convex function of the investment-capital ratio, I/K.

EXAMPLE 1 Let $J = G(I, K) = \frac{1}{2}\beta I^2/K$, where $\beta > 0$. Then G is homogeneous of degree one w.r.t. I and K and gives $J/K = \frac{1}{2}\beta(I/K)^2 \equiv g(I/K)$. □

A further important property of (14.22) is that the cash-flow function in (14.3) becomes homogeneous of degree one w.r.t. K, L, and I in the "normal" case where the production function has CRS. This has two implications. First, Hayashi's theorem applies (see below). Second, the q-theory can easily be incorporated into a model of economic growth.[6]

Does the hypothesis of linear homogeneity of the cash flow in K, L, and I make economic sense? According to the replication argument it does. Suppose a given firm has K units of installed capital and produces Y units of output with L units of labor. When at the same time the firm invests I units of account in new capital, it obtains the cash flow R after deducting the installation costs, $G(I, K)$. Then it makes sense to assume that the firm could do the same thing at another place, hereby doubling its cash-flow. (Of course, owing to the possibility of indivisibilities, this reasoning does not take us all the way to linear homogeneity. Moreover, the argument ignores that also land is a necessary input. As discussed in Chapter 2, the empirical evidence on linear homogeneity is mixed.)

In view of (i) of Lemma 1, the linear homogeneity assumption for G allows us to write (14.19) as

$$g'(I/K) = q - 1. \tag{14.23}$$

This equation defines the investment-capital ratio, I/K , as an implicit function, m, of q :

$$\frac{I_t}{K_t} = m(q_t), \quad \text{where} \quad m(1) = 0 \quad \text{and} \quad m' = \frac{1}{g''} > 0, \tag{14.24}$$

by implicit differentiation in (14.23). In this case q encompasses all information that is of relevance to the decision about the investment-capital ratio.

In Example 1 above we have $g(I/K) = \frac{1}{2}\beta(I/K)^2$, in which case (14.23) gives $I/K = (q-1)/\beta$. So in this case we have $m(q) = q/\beta - 1/\beta$, a linear investment function, as illustrated in Fig. 14.3. The parameter β can be interpreted as the degree of sluggishness in the capital adjustment. The degree of sluggishness reflects the degree of convexity of installation costs.[7] The stippled lines in Fig. 14.3 are explained below. Generally the graph of the investment function is positively sloped, but not necessarily linear.

To see how the shadow price q changes over time along the optimal path, we rearrange (14.11):

$$\dot{q}_t = (r_t + \delta)q_t - F_K(K_t, L_t) + G_K(I_t, K_t). \tag{14.25}$$

[6] The relationship between the function g and other ways of formulating the theory is commented on in Appendix C.

[7] For a twice differentiable function, $f(x)$, with $f'(x) \neq 0$, we define the *degree of convexity* in the point x by $f''(x)/f'(x)$. So the degree of convexity of $g(I/K)$ is $g''/g' = (I/K)^{-1} = \beta(q-1)^{-1}$ and thereby we have $\beta = (q-1)g''/g'$. So, for given q, the degree of sluggishness is proportional to the degree of convexity of adjustment costs.

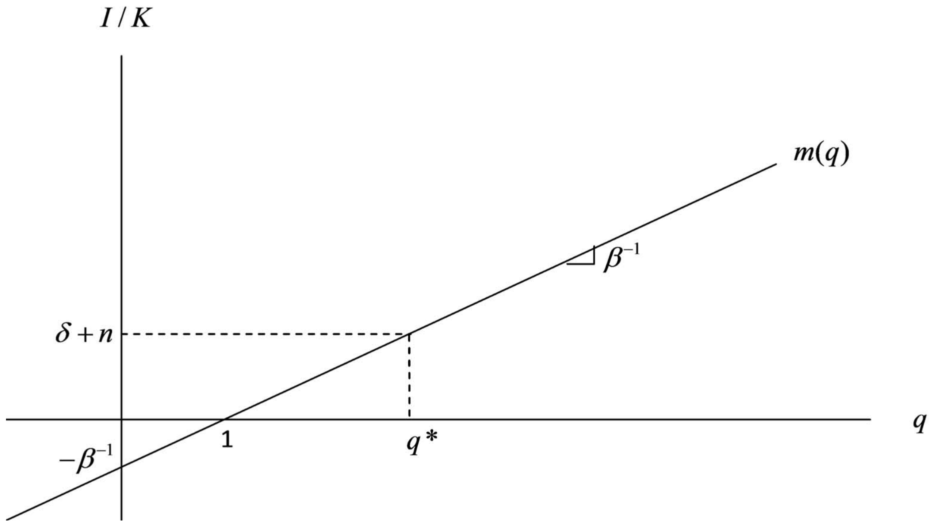

Figure 14.3: Optimal investment-capital ratio as a function of the shadow price of installed capital when $g(I/K) = \frac{1}{2}\beta(I/K)^2$.

Recall that $-G_K(I_t, K_t)$ indicates how much *lower* the installation costs are as a result of the marginal unit of installed capital. In the special case (14.22) we have from Lemma 1

$$G_K(I, K) = g(\frac{I}{K}) - g'(\frac{I}{K})\frac{I}{K} = g(m(q)) - (q-1)m(q),$$

using (14.24) and (14.23).

Inserting this into (14.25) gives

$$\dot{q}_t = (r_t + \delta)q_t - F_K(K_t, L_t) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{14.26}$$

This differential equation is very useful in macroeconomic analysis, as we will soon see, cf. Fig. 14.4 below.

In a macroeconomic context, for steady state to achievable, gross investment must be large enough to match not only capital depreciation, but also growth in the labor input. Otherwise a constant capital-labor ratio can not be sustained. That is, the investment-capital ratio, I/K, must be equal to the sum of the depreciation rate and the growth rate of the labor force, i.e., $\delta + n$. The level of q which is required to motivate such an investment-capital ratio is called q^* in Fig. 14.3.

14.2 Marginal q and average q

Our q above, determining investment, should be distinguished from what is usually called Tobin's q or average q. In a more general context, let p_{It} denote the current purchase price (in terms of output units) per unit of the investment good (before installment). Then *Tobin's* q or *average* q, q_t^a, is defined as $q_t^a \equiv V_t/(p_{It}K_t)$, that is, Tobin's q is the ratio of the market value of the firm to the replacement value of the firm in the sense of the "reacquisition value of the capital goods before installment costs" (the top index "a" stands for "average"). In our simplified context we have $p_{It} \equiv 1$ (the price of the investment good is the same as that of the output good). Therefore Tobin's q can be written

$$q_t^a \equiv \frac{V_t}{K_t} = \frac{V^*(K_t, t)}{K_t}, \tag{14.27}$$

where the equality holds for an optimizing firm. Conceptually this is different from the firm's internal shadow price on capital, i.e., what we have denoted q_t in the previous sections. In the language of the q-theory of investment this q_t is the *marginal* q, representing the value to the firm of one *extra* unit of installed capital relative to the price of un-installed capital equipment. The term marginal q is natural since along the optimal path, as a slight generalization of (14.18), we must have $q_t = (\partial V^*/\partial K_t)/p_{It}$. Letting q_t^m ("m" for "marginal") be an alternative symbol for this q_t, we have in our model above, where we consider the special case $p_{It} \equiv 1$,

$$q_t^m \equiv q_t = \frac{\partial V^*}{\partial K_t}. \tag{14.28}$$

The two concepts, average q and marginal q, have not always been clearly distinguished in the literature. What is directly relevant to the investment decision is marginal q. Indeed, the analysis above showed that optimal investment is an increasing function of q^m. Further, the analysis showed that a "critical" value of q^m is 1 and that only if $q^m > 1$, is positive gross investment warranted.

The importance of q^a is that it can be measured empirically as the ratio of the sum of the share market value of the firm and its debt to the current acquisition value of its total capital before installment. Since q^m is much harder to measure than q^a, it is important to know the relationship between q^m and q^a. Fortunately, we have a simple theorem giving conditions under which $q^m = q^a$.

THEOREM (Hayashi, 1982) Assume the firm is a price taker, that the production function F is jointly concave in (K, L), and that the installation cost function G is jointly convex in (I, K).[8] Then, along an optimal path we have:

[8] That is, in addition to (14.2), we assume $G_{KK} \geq 0$ and $G_{II}G_{KK} - G_{IK}^2 \geq 0$. The specification in Example 1 above satisfies this.

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in $(K,\, L)$ and/or G is strictly convex in $(I,\, K)$.

Proof. See Appendix D.

The assumption that the firm is a price taker may, of course, seem critical. The Hayashi theorem has been generalized, however. Also a monopolistic firm, facing a downward-sloping demand curve and setting its own price, may have a cash flow which is homogeneous of degree one in the three variables K, L, and I. If so, then the condition $q_t^m = q_t^a$ for all $t \geq 0$ still holds (Abel 1990). Abel and Eberly (1994) present further generalizations.

In any case, when q^m is approximately equal to (or just proportional to) q^a, the theory gives a remarkably simple operational investment function, $I = m(q^a)K$, cf. (14.24). At the macro level we interpret q^a as the market valuation of the firms relative to the replacement value of their total capital stock. This market valuation is an indicator of the expected future earnings potential of the firms. Under the conditions in (i) of the Hayashi theorem the market valuation also indicates the marginal earnings potential of the firms, hence, it becomes a determinant of their investment. This establishment of a relationship between the stock market and firms' aggregate investment is the basic point in Tobin (1969).

14.3 Applications

Capital installation costs in a closed economy

Allowing for convex capital installation costs in the economy has far-reaching implications for the causal structure of a model of a closed economy. Investment decisions attain an active role in the economy and forward-looking expectations become important for these decisions. Expected future market conditions and announced future changes in corporate taxes and depreciation allowance will affect firms' investment already today.

The essence of the matter is that current and expected future interest rates have to adjust for aggregate saving to equal aggregate investment, that is, for the output and asset markets to clear. Given full employment ($L_t = \bar{L}_t$), the output market clears when

$$F(K_t, \bar{L}_t) - G(I_t, K_t) = \text{value added} \equiv GDP_t = C_t + I_t,$$

where C_t is determined by the intertemporal utility maximization of the forward-looking households, and I_t is determined by the intertemporal value maximization of the forward-looking firms facing strictly convex installation costs. Like in the determination of C_t, current and expected future interest rates now also matter for the determination of I_t. This is the first time in this book where *clearing in the output market* is assigned an *active* role. In the earlier models investment was just a passive reflection of household saving. Desired investment was automatically equal to the residual of national income left over after consumption decisions had taken place. Nothing had to adjust to clear the output market, neither interest rates nor output. In contrast, in the present framework adjustments in interest rates and/or the output level are needed for the continuous clearing in the output market and these adjustments are decisive for the macroeconomic dynamics.

In actual economies there may of course exist "secondary markets" for used capital goods and markets for renting capital goods owned by others. In view of installation costs and similar, however, shifting capital goods from one plant to another is generally costly. Therefore the turnover in that kind of markets tends to be limited and there is little underpinning for the earlier models' supposition that the current interest rate should be tied down by a requirement that such markets clear.

In for instance Abel and Blanchard (1983) a Ramsey-style model integrating the q-theory of investment is presented. The authors study the two-dimensional general equilibrium dynamics resulting from the adjustment of current and expected future (short-term) interest rates *needed for the output market to clear*. Adjustments of the whole structure of interest rates (the yield curve) take place and constitute the equilibrating mechanism in the output and asset markets.

By having output market equilibrium playing this role in the model, a first step is taken towards medium- and short-run macroeconomic theory. We take further steps in later chapters, by allowing imperfect competition and nominal price rigidities to enter the picture. Then the demand side gets an active role both in the determination of q (and thereby investment) and in the determination of aggregate output and employment. This is what Keynesian theory (old and new) deals with.

In the remainder of this chapter we will still assume perfect competition in all markets including the labor market. In this sense we will stay within the neoclassical framework (supply-dominated models) where, by instantaneous adjustment of the real wage, labor demand continuously matches labor supply. The next two subsections present examples of how Tobin's q-theory of investment can be integrated into the neoclassical framework. To avoid the more complex dynamics arising in a closed economy, we shift the focus to a small open economy. This allows concentrating on a dynamic system with an exogenous interest rate.

A small open economy with capital installation costs

By introducing convex capital installation costs in a model of a small open economy (SOE), we avoid the counterfactual outcome that the capital stock adjusts *instantaneously* when the interest rate in the world financial market changes. In the standard neoclassical growth model for a small open economy, without convex capital installation costs, a rise in the interest rate leads immediately to a complete adjustment of the capital stock so as to equalize the net marginal productivity of capital to the new higher interest rate. Moreover, in that model expected *future* changes in the interest rate or in corporate taxes and depreciation allowances do *not* trigger an investment response until these changes actually happen. In contrast, when convex installation costs are present, expected future changes tend to influence firms' investment already today.

We assume:

1. Perfect mobility across borders of goods and financial capital.
2. Domestic and foreign financial claims are perfect substitutes.
3. No mobility across borders of labor.
4. Labor supply is inelastic and constant and there is no technological progress.
5. The capital installation cost function $G(I, K)$ is homogeneous of degree 1.

In this setting the SOE faces an exogenous interest rate, r, given from the world financial market. We assume r is a positive constant. The aggregate production function, $F(K, L)$, is neoclassical and concave as in the previous sections. With $\bar{L} > 0$ denoting the constant labor supply, continuous clearing in the labor market under perfect competition gives $L_t = \bar{L}$ for all $t \geq 0$ and

$$w_t = F_L(K_t, \bar{L}) \equiv w(K_t). \tag{14.29}$$

At any time t, K_t is predetermined in the sense that due to the convex installation costs, changes in K take time. Thus (14.29) *determines* the market real wage w_t.

To pin down the evolution of the economy, we now derive two coupled differential equations in K and q. Inserting (14.24) into (14.6) gives

$$\dot{K}_t = (m(q_t) - \delta)K_t, \qquad K_0 > 0 \text{ given.} \tag{14.30}$$

As to the dynamics of q, we have (14.26). Since the capital installation cost function $G(I, K)$ is assumed to be homogeneous of degree 1, point (iii) of Lemma 1 applies and we can write (14.26) as

$$\dot{q}_t = (r + \delta)q_t - F_K(K_t, \bar{L}) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{14.31}$$

As r and $\bar{L}$ are exogenous, the capital stock, K, and its shadow price, q, are the only endogenous variables in the differential equations (14.30) and (14.31).

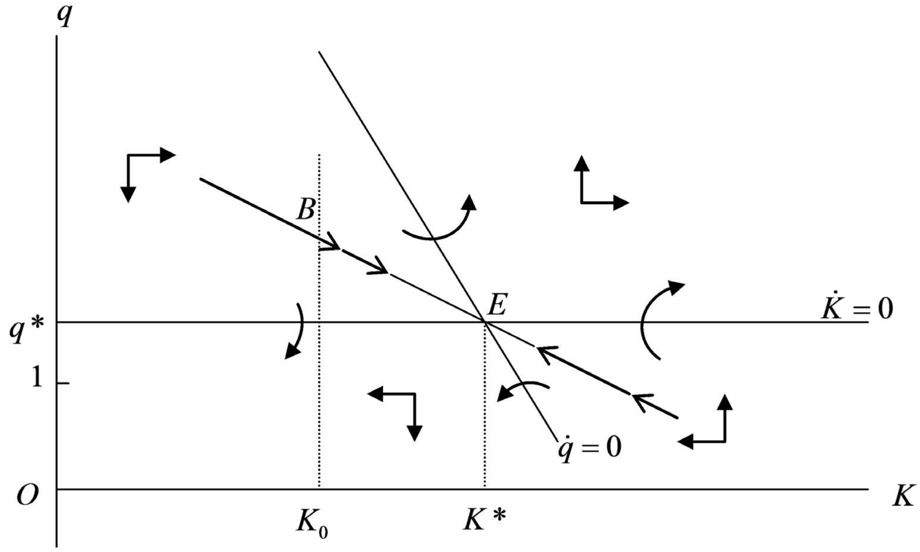

Figure 14.4: Phase diagram for investment dynamics in a small open economy (a case where $\delta > 0$).

In addition, we have an initial condition for K and a necessary transversality condition involving q, namely

$$\lim_{t\to\infty} K_t q_t e^{-rt} = 0. \tag{14.32}$$

Fig. 14.4 shows the phase diagram for these two coupled differential equations. Let q^* be defined as the value of q satisfying the equation $m(q) = \delta$. Since $m' > 0$, q^* is unique. Suppressing for convenience the explicit time subscripts, we then have

$$\dot{K} = 0 \text{ for } m(q) = \delta, \text{ i.e., for } q = q^*.$$

As $\delta > 0$, we have $q^* > 1$. This is so because also mere reinvestment to offset capital depreciation requires an incentive, namely that the marginal value to the firm of replacing worn-out capital is larger than the purchase price of the investment good (since the installation cost must also be compensated). From (14.30) is seen that

$$\dot{K} \gtreqless 0 \text{ for } m(q) \gtreqless \delta, \text{ respectively, i.e., for } q \gtreqless q^*, \text{ respectively,}$$

cf. the horizontal arrows in Fig. 14.4.

From (14.31) we have

$$\dot{q} = 0 \text{ for } 0 = (r+\delta)q - F_K(K, \bar{L}) + g(m(q)) - (q-1)m(q). \tag{14.33}$$

If, in addition $\dot{K} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta$), this gives

$$0 = (r + \delta)q^* - F_K(K, \bar{L}) + g(\delta) - (q^* - 1)\delta, \tag{14.34}$$

where the right-hand-side is increasing in K, in view of $F_{KK} < 0$. Hence, there exists at most one value of K such that the steady state condition (14.34) is satisfied;[9] this value is denoted K^*, corresponding to the steady state point E in Fig. 14.4. The question is now: what is the slope of the $\dot{q} = 0$ locus? In Appendix E it is shown that at least in a neighborhood of the steady state point E, this slope is negative in view of the assumption $r > 0$ and $F_{KK} < 0$. From (14.31) we see that

$$\dot{q} \lesseqgtr 0 \text{ for points to the left and to the right, respectively, of the } \dot{q} = 0 \text{ locus,}$$

since $F_{KK}(K_t, \bar{L}) < 0$. The vertical arrows in Fig. 14.4 show these directions of movement.

Altogether the phase diagram shows that the steady state E is a saddle point, and since there is one predetermined variable, K, and one jump variable, q, and the saddle path is not parallel to the jump variable axis, the steady state is saddle-point stable. At time 0 the economy will be at the point B in Fig. 14.4 where the vertical line $K = K_0$ crosses the saddle path. Then the economy will move along the saddle path towards the steady state. This solution satisfies the transversality condition (14.32) and is the unique solution to the model (for details, see Appendix F).

The effect of an unanticipated rise in the interest rate Suppose that until time 0 the economy has been in the steady state E in Fig. 14.4. Then, an unexpected shift in the interest rate occurs so that the new interest rate is a constant $r' > r$. We assume that the new interest rate is rightly expected to remain at this level forever. From (14.30) we see that q^* is not affected by this shift, hence, the $\dot{K} = 0$ locus is not affected. However, (14.33) implies that the $\dot{q} = 0$ locus and K^* shift to the left, in view of $F_{KK}(K, \bar{L}) < 0$.

Fig. 14.5 illustrates the situation for $t > 0$. At time $t = 0$ the shadow price q jumps down to a level corresponding to the point B in Fig. 14.5. There is now a heavier discounting of the future benefits that the marginal unit of capital can provide. As a result the incentive to invest is diminished and gross investment will not even compensate for the depreciation of capital. Hence, the capital stock decreases gradually. This is where we see a crucial role of convex capital installation costs in an open economy. For now, the installation costs are the costs

[9] And assuming that F satisfies the Inada conditions, we are sure that such a value exists since (14.34) gives $F_K(K, \bar{L}) = rq^* + g(\delta) + \delta > 0$.

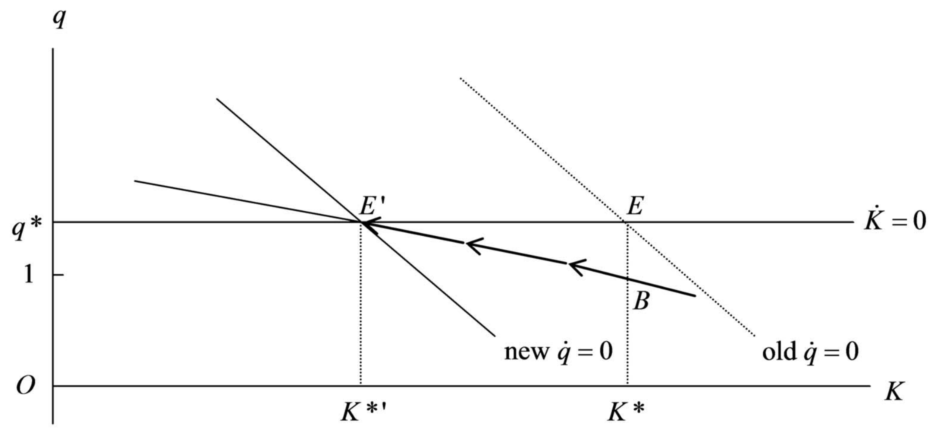

Figure 14.5: Phase portrait of an unanticipated rise in r (the case $\delta > 0$).

associated with disinvestment (dismantling and selling out of machines). If these convex costs were not present, we would get the same counterfactual prediction as from the previous open-economy models in this book, namely that the new steady state is attained immediately after the shift in the interest rate.

As the capital stock is diminished, the marginal productivity of capital rises and so does q. The economy moves along the new saddle path and approaches the new steady state E' as time goes by.

Suppose that for some reason such a decrease in the capital stock is not desirable from a social point of view; this could be because of positive external effects of capital and investment, e.g., a kind of "learning by doing". Then the government could decide to implement an investment subsidy σ, $0 < \sigma < 1$, so that to attain an investment level I, purchasing the investment goods involves a cost of $(1-\sigma)I$. Assuming the subsidy is financed by some tax not affecting firms' behavior (for example a constant tax on households' consumption), investment is increased again and the economy may in the long run end up at the old steady-state level of K (but the new q^* will be lower than the old).

A growing small open economy with capital installation costs*

The basic assumptions are the same as in the previous section except that now labor supply, $\bar{L}_t$, grows at the constant rate $n \geq 0$, while the technology level, T, grows at the constant rate $\gamma \geq 0$ (both rates exogenous and constant) and the production function is neoclassical with CRS. We assume that the world market real interest rate, r, is a constant and satisfies $r > \gamma + n$. Still assuming full employment, we have $L_t = \bar{L}_t = \bar{L}_0 e^{nt}$.

In this setting the production function on intensive form is useful:

$$Y = F(K, T\bar{L}) = F(\frac{K}{T\bar{L}}, 1)T\bar{L} \equiv f(\tilde{k})T\bar{L},$$

where $\tilde{k} \equiv K/(T\bar{L})$ and f satisfies $f' > 0$ and $f'' < 0$. Still assuming perfect competition, the market-clearing real wage at time t is determined as

$$w_t = F_2(K_t, T_t\bar{L}_t)T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right] T_t \equiv \tilde{w}(\tilde{k}_t)T_t,$$

where both $\tilde{k}_t$ and T_t are predetermined. By log-differentiation of $\tilde{k} \equiv K/(T\bar{L})$ w.r.t. time we get $\dot{\tilde{k}}_t/\tilde{k}_t = \dot{K}_t/K_t - (\gamma + n)$. Substituting (14.30), we get

$$\dot{\tilde{k}}_t = [m(q_t) - (\delta + \gamma + n)]\, \tilde{k}_t. \tag{14.35}$$

The change in the shadow price of capital is now described by

$$\dot{q}_t = (r + \delta)q_t - f'(\tilde{k}_t) + g(m(q_t)) - (q_t - 1)m(q_t), \tag{14.36}$$

from (14.26). In addition, the transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t q_t e^{-(r-\gamma-n)t} = 0, \tag{14.37}$$

must hold.

The differential equations (14.35) and (14.36) constitute our new dynamic system. Fig. 14.6 shows the phase diagram, which is qualitatively similar to that in Fig. 14.4. We have

$$\dot{\tilde{k}} = 0 \quad \text{for} \;\; m(q) = \delta + \gamma + n, \;\text{ i.e., for } q = q^*,$$

where q^* now is defined by the requirement $m(q^*) = \delta + \gamma + n$. Notice, that when $\gamma + n > 0$, we get a larger steady state value q^* than in the previous section. This is so because now a higher investment-capital ratio is required for a steady state to be possible. Moreover, the transversality condition (14.12) is satisfied in the steady state.

From (14.36) we see that $\dot{q} = 0$ now requires

$$0 = (r + \delta)q - f'(\tilde{k}) + g(m(q)) - (q - 1)m(q).$$

If, in addition $\dot{\tilde{k}} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta + \gamma + n$), this gives

$$0 = (r + \delta)q^* - f'(\tilde{k}) + g(\delta + \gamma + n) - (q^* - 1)(\delta + \gamma + n).$$

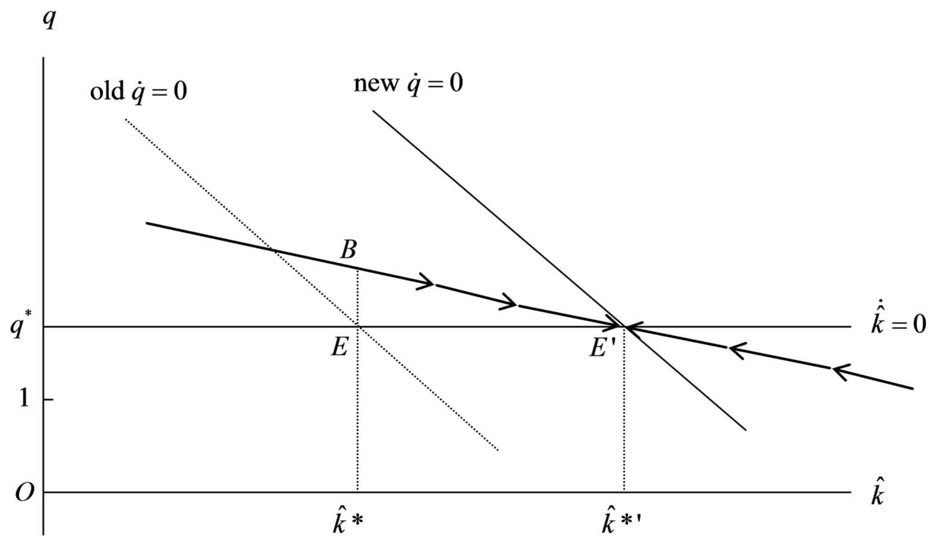

Figure 14.6: Phase portrait of an unanticipated fall in r (a growing economy with $\delta + \gamma + n \geq \gamma + n > 0$).

Here, the right-hand-side is increasing in $\tilde{k}$ (in view of $f''(\tilde{k}) < 0$). Hence, the steady state value $\tilde{k}^*$ of the effective capital-labor ratio is unique, cf. the steady state point E in Fig. 14.6.

By the assumption $r > \gamma + n$ we have, at least in a neighborhood of E in Fig. 14.6, that the $\dot{q} = 0$ locus is negatively sloped (see Appendix E).[10] Again the steady state is a saddle point, and the economy moves along the saddle path towards the steady state.

In Fig. 14.6 it is assumed that until time 0, the economy has been in the steady state E. Then, an unexpected shift in the interest rate to a *lower* constant level, r', takes place. The $\dot{q} = 0$ locus is shifted to the right, in view of $f'' < 0$. The shadow price, q, immediately jumps up to a level corresponding to the point B in Fig. 14.6. The economy moves along the new saddle path and approaches the new steady state E' with a higher effective capital-labor ratio as time goes by. In Exercise 14.2 the reader is asked to examine the analogue situation where an unanticipated downward shift in the rate of technological progress takes place.

[10] In our perfect foresight model we in fact *have* to assume $r > \gamma + n$ for the firm's maximization problem to be well-defined. If instead $r \leq \gamma + n$, the market value of the representative firm would be infinite, and maximization would loose its meaning.

14.4 Concluding remarks

Tobin's q-theory of investment gives a remarkably simple operational macroeconomic investment function, in which the key variable explaining aggregate investment is the valuation of the firms by the stock market relative to the replacement value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

When faced with strictly convex installation costs, the firm has to take the *future* into account to invest optimally. Therefore, the firm's *expectations* become important. Owing to the strictly convex installation costs, the firm adjusts its capital stock only *gradually* when new information arises. This investment smoothing is analogue to consumption smoothing.

By incorporating these features, Tobin's q-theory helps explaining the sluggishness in investment we see in the empirical data. And the theory avoids the counterfactual outcome from earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world market changes. So the theory takes into account the time lags in capital adjustment in real life, a feature which may, perhaps, be abstracted from in long-run analysis and models of economic growth, but not in short- and medium-run analysis.

Many econometric tests of the q theory of investment have been made, often with quite critical implications. Movements in q^a, even taking account of changes in taxation, seemed capable of explaining only a minor fraction of the movements in investment. And the estimated equations relating fixed capital investment to q^a typically give strong auto-correlation in the residuals. Other variables, in particular availability of current corporate profits for internal financing, seem to have explanatory power independently of q^a (see Abel 1990, Chirinko 1993, Gilchrist and Himmelberg, 1995). So there is reason to be somewhat sceptical towards the notion that all information of relevance for the investment decision is reflected by the market valuation of firms. This throws doubt on the basic assumption in Hayashi's theorem or its generalization, the assumption that firms' cash flow tends to be homogeneous of degree one w.r.t. K, L, and I.

Going outside the model, there are further circumstances relaxing the link between q^a and investment. In the real world with many production sectors, physical capital is heterogeneous. If for example a sharp unexpected rise in the price of energy takes place, a firm with energy-intensive technology will loose in market value. At the same time it has an incentive to invest in energy-saving capital equipment. Hence, we might observe a fall in q^a at the same time as investment increases.

Imperfections in credit markets are ignored by the model. Their presence further loosens the relationship between q^a and investment and may help explain the observed positive correlation between investment and corporate profits.

We might also question that capital installation costs really have the hypothesized *strictly convex* form. It is one thing that there are costs associated with installation, reorganizing and retraining etc., when new capital equipment is procured. But should we expect these costs to be strictly convex in the volume of investment? To think about this, let us for a moment ignore the role of the existing capital stock. Hence, we write total installation costs $J = G(I)$ with $G(0) = 0$. It does not seem problematic to assume $G'(I) > 0$ for $I > 0$. The question concerns the assumption $G''(I) > 0$. According to this assumption the average installation cost $G(I)/I$ must be increasing in I.[11] But against this speaks the fact that capital installation may involve indivisibilities, fixed costs, acquisition of new information etc. All these features tend to imply *decreasing* average costs. In any case, at least at the microeconomic level one should expect unevenness in the capital adjustment process rather than the above smooth adjustment.

Because of the mixed empirical success of the convex installation cost hypothesis other theoretical approaches that can account for sluggish and sometimes non-smooth and lumpy capital adjustment have been considered: uncertainty, investment irreversibility, indivisibility, or financial problems due to bankruptcy costs (Nickell 1978, Zeira 1987, Dixit and Pindyck 1994, Caballero 1999, Adda and Cooper 2003). These approaches notwithstanding, it turns out that the q-theory of investment has recently been somewhat rehabilitated from both a theoretical and an empirical point of view. At the theoretical level Wang and Wen (2010) show that financial frictions in the form of collateralized borrowing at the firm level can give rise to strictly convex adjustment costs at the aggregate level yet at the same time generate lumpiness in plant-level investment. For large firms, unlikely to be much affected by financial frictions, Eberly et al. (2008) find that the theory does a good job in explaining investment behavior.

In any case, the q-theory of investment is in different versions widely used in short- and medium-run macroeconomics because of its simplicity and the appealing link it establishes between asset markets and firms' investment. And the q-theory has also had an important role in studies of the housing market and the role of housing prices for household wealth and consumption, a theme to which we return in the next chapter.

[11] Indeed, for $I \neq 0$ we have $d[G(I)/I]/dI = [IG'(I) - G(I)]/I^2 > 0$, when G is strictly convex ($G'' > 0$) and $G(0) = 0$.

14.5 Literature notes

A first sketch of the q-theory of investment is contained in Tobin (1969). Later advances of the theory took place through the contributions of Hayashi (1982) and Abel (1990).

Both the Ramsey model and the Blanchard OLG model for a closed market economy may be extended by adding strictly convex capital installation costs, see Abel and Blanchard (1983) and Lim and Weil (2003). Adding a public sector, such a framework is useful for the study of how different subsidies, taxes, and depreciation allowance schemes affect investment in physical capital as well as housing, see, e.g., Summers (1981), Abel and Blanchard (1983), and Dixit (1990).

Groth and Madsen (2013) study medium-term fluctuations arising in a Ramsey-Tobin's q framework when extended by sluggishness in real wage adjustments.

14.6 Appendix

A. When value maximization is - and is not - equivalent with continuous static profit maximization

For the idealized case where tax distortions, asymmetric information, and problems with enforceability of financial contracts are absent, the Modigliani-Miller theorem (Modigliani and Miller, 1958) says that the financial structure of the firm is both indeterminate and irrelevant for production outcomes. Considering the firm described in Section 14.1, the implied separation of the financing decision from the production and investment decision can be exposed in the following way.

Simple version of the Modigliani-Miller theorem Although the theorem allows for risk, we here ignore risk. Let the real debt of the firm be denoted B_t and the real dividends, X_t. We then have the accounting relationship

$$\dot{B}_t = X_t - (F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t).$$

A positive X_t represents dividends in the usual meaning (payout to the owners of the firm), whereas a negative X_t can be interpreted as emission of new shares of stock. Since we assume perfect competition, the time path of w_t and r_t is exogenous to the firm.

We first consider the firm's combined financing and production-investment problem, which we call *Problem I*. We assume that those who own the firm at time 0 want it to maximize its net worth, i.e., the present value of expected future dividends:

$$
\begin{aligned}
\max_{(L_t, I_t, X_t)_{t=0}^{\infty}} \tilde{V}_0 &= \int_0^{\infty} X_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.} \\
L_t &\geq 0, I_t \text{ free,} \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given,} \quad K_t \geq 0 \text{ for all } t, \\
\dot{B}_t &= X_t - \left(F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t\right), \\
& \quad \text{where } B_0 \text{ is given,} \qquad (14.38) \\
\lim_{t\to\infty} B_t e^{-\int_0^t r_s ds} &\leq 0. \qquad \text{(NPG)}
\end{aligned}
$$

The last constraint is a No-Ponzi-Game condition, saying that a positive debt should in the long run at most grow at a rate which is *less* than the interest rate.

In Section 14.1 we considered another problem, namely a separate investment-production problem:

$$
\begin{aligned}
\max_{(L_t, I_t)_{t=0}^{\infty}} V_0 &= \int_0^{\infty} R_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.,} \\
R_t &\equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \\
L_t &\geq 0, I_t \text{ free,} \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given,} \quad K_t \geq 0 \text{ for all } t.
\end{aligned}
$$

Let this problem, where the financing aspects are ignored, be called *Problem II.* When considering the relationship between Problem I and Problem II, the following mathematical fact is useful.

LEMMA A1 Consider a continuous function $a(t)$ and a differentiable function $f(t)$. Then

$$\int_{t_0}^{t_1} (f'(t) - a(t)f(t)) e^{-\int_{t_0}^{t} a(s)ds} dt = f(t_1) e^{-\int_{t_0}^{t_1} a(s)ds} - f(t_0).$$

Proof. Integration by parts from time t_0 to time t_1 yields

$$\int_{t_0}^{t_1} f'(t) e^{-\int_{t_0}^{t} a(s)ds} dt = f(t) e^{-\int_{t_0}^{t} a(s)ds} \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} f(t) a(t) e^{-\int_{t_0}^{t} a(s)ds} dt.$$

Hence,

$$
\begin{aligned}
& \int_{t_0}^{t_1} (f'(t) - a(t)f(t)) e^{-\int_{t_0}^{t} a(s)ds} dt \\
= \; & f(t_1) e^{-\int_{t_0}^{t_1} a(s)ds} - f(t_0). \qquad \square
\end{aligned}
$$

CLAIM 1 If $(K_t^*, B_t^*, L_t^*, I_t^*, X_t^*)_{t=0}^{\infty}$ is a solution to Problem I, then $(K_t^*, L_t^*, I_t^*)_{t=0}^{\infty}$ is a solution to Problem II.

Proof. By (14.38) and the definition of R_t, $X_t = R_t + \dot{B}_t - r_t B_t$ so that

$$\tilde{V}_0 = \int_0^{\infty} X_t e^{-\int_0^t r_s ds} dt = V_0 + \int_0^{\infty} (\dot{B}_t - r_t B_t) e^{-\int_0^t r_s ds} dt. \tag{14.39}$$

In Lemma A1, let $f(t) = B_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty} \int_0^T (\dot{B}_t - r_t B_t) e^{-\int_0^t r_s ds} dt = \lim_{T\to\infty} B_T e^{-\int_0^T r_s ds} - B_0 \leq -B_0,$$

where the weak inequality is due to (NPG). Substituting this into (14.39), we see that maximum of net worth $\tilde{V}_0$ is obtained by maximizing V_0 and ensuring $\lim_{T\to\infty} B_T e^{-\int_0^T r_s ds} = 0$, in which case net worth equals ((maximized V_0) $- B_0$), where B_0 is given. So a plan that maximizes net worth of the firm must also maximize V_0 in Problem II. □

Consequently it does not matter for the firm's production and investment behavior whether the firm's investment is financed by issuing new debt or by issuing shares of stock. Moreover, if we assume investors do not care about whether they receive the firm's earnings in the form of dividends or valuation gains on the shares, the firm's dividend policy is also irrelevant. Hence, from now on we can concentrate on the investment-production problem, Problem II above.

The case with no capital installation costs Suppose the firm has no capital installation costs. Then the cash flow reduces to $R_t = F(K_t, L_t) - w_t L_t - I_t$.

CLAIM 2 When there are no capital installation costs, Problem II can be reduced to a series of static profit maximization problems.

Proof. Current (pure) profit is defined as

$$\Pi_t = F(K_t, L_t) - w_t L_t - (r_t + \delta) K_t \equiv \Pi(K_t, L_t).$$

It follows that R_t can be written

$$R_t = F(K_t, L_t) - w_t L_t - (\dot{K}_t + \delta K_t) = \Pi_t + (r_t + \delta) K_t - (\dot{K}_t + \delta K_t). \tag{14.40}$$

Hence,

$$V_0 = \int_0^{\infty} \Pi_t e^{-\int_0^t r_s ds} dt + \int_0^{\infty} (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt. \tag{14.41}$$

The first integral on the right-hand side of this expression is independent of the second. Indeed, the firm can maximize the first integral by *renting* capital and labor, K_t and L_t, at the going factor prices, $r_t + \delta$ and w_t, respectively, such that $\Pi_t = \Pi(K_t, L_t)$ is maximized at each t. The factor costs are accounted for in the definition of Π_t.

The second integral on the right-hand side of (14.41) is the present value of net revenue from renting capital out to others. In Lemma A1, let $f(t) = K_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty} \int_0^T (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt = K_0 - \lim_{T\to\infty} K_T e^{-\int_0^T r_s ds} = K_0, \tag{14.42}$$

where the last equality comes from the fact that maximization of V_0 requires maximization of the left-hand side of (14.42) which in turn, since K_0 is given, requires minimization of $\lim_{T\to\infty} K_T e^{-\int_0^T r_s ds}$. The latter expression is always non-negative and can be made zero by choosing any time path for K_t such that $\lim_{T\to\infty} K_T = 0$. (We may alternatively put it this way: it never pays the firm to accumulate costly capital so fast in the long run that $\lim_{T\to\infty} K_T e^{-\int_0^T r_s ds} > 0$, that is, to maintain accumulation of capital at a rate equal to or higher than the interest rate.) Substituting (14.42) into (14.41), we get $V_0 = \int_0^\infty \Pi_t e^{-\int_0^t r_s ds} dt + K_0$.

The conclusion is that, given K_0,[12] V_0 is maximized if and only if K_t and L_t are at each t chosen such that $\Pi_t = \Pi(K_t, L_t)$ is maximized. $\square$

The case with strictly convex capital installation costs Now we reintroduce the capital installation cost function $G(I_t, K_t)$, satisfying in particular the condition $G_{II}(I, K) > 0$ for all (I, K). Then, as shown in the text, the firm adjusts to a change in its environment, say a downward shift in r, by a *gradual* adjustment of K, in this case upward, rather than attempting an instantaneous maximization of $\Pi(K_t, L_t)$. The latter would entail an instantaneous upward jump in K_t of size $\Delta K_t = a > 0$, requiring $I_t \cdot \Delta t = a$ for $\Delta t = 0$. This would require $I_t = \infty$, which implies $G(I_t, K_t) = \infty$, which may interpreted either as such a jump being impossible or at least so costly that no firm will pursue it.

[12] Note that in the absence of capital installation costs, the historically given K_0 is no more "given" than the firm may instantly let it jump to a lower or higher level. In the first case the firm would immediately sell a bunch of its machines and in the latter case it would immediately buy a bunch of machines. Indeed, without convex capital installation costs nothing rules out jumps in the capital stock. But such jumps just reflect an immediate jump, in the opposite direction, in another asset item in the balance sheet and leave the maximized net worth of the firm unchanged.

Proof that q_t satisfies (14.15) along an interior optimal path Rearranging (14.11) and multiplying through by the integrating factor $e^{-\int_0^t (r_s+\delta)ds}$, we get

$$[(r_t+\delta)q_t - \dot{q}_t]\, e^{-\int_0^t (r_s+\delta)ds} = (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds}, \tag{14.43}$$

where $F_{Kt} \equiv F_K(K_t, L_t)$ and $G_{Kt} \equiv G_K(I_t, K_t)$. In Lemma A1, let $f(t) = q_t$, $a(t) = r_t + \delta$, $t_0 = 0$, $t_1 = T$. Then

$$\begin{aligned} \int_0^T [(r_t+\delta)q_t - \dot{q}_t]\, e^{-\int_0^t (r_s+\delta)ds} dt &= q_0 - q_T e^{-\int_0^T (r_s+\delta)ds} \\ &= \int_0^T (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds} dt, \end{aligned}$$

where the last equality comes from (14.43). Letting $T \to \infty$, we get

$$q_0 - \lim_{T\to\infty} q_T e^{-\int_0^T (r_s+\delta)ds} = q_0 = \int_0^\infty (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds} dt, \tag{14.44}$$

where the first equality follows from the transversality condition (14.14), which we repeat here:

$$\lim_{t\to\infty} q_t e^{-\int_0^t r_s ds} = 0. \tag{*}$$

Indeed, since $\delta \geq 0$, $\lim_{T\to\infty}(e^{-\int_0^T r_s ds} e^{-\delta T}) = 0$, when (*) holds. Initial time is arbitrary, and so we may replace 0 and t in (14.44) by t and τ, respectively. The conclusion is that (14.15) holds along an interior optimal path, given the transversality condition (*). A proof of necessity of the transversality condition (*) is given in Appendix B.[13]

B. Transversality conditions

In view of (14.44), a qualified conjecture is that the condition $\lim_{t\to\infty} q_t e^{-\int_0^t (r_s+\delta)ds} = 0$ is necessary for optimality. This is indeed true, since this condition follows from the stronger transversality condition (*) in Appendix A, the necessity of which along an optimal path we will now prove.

Proof of necessity of (14.14) As the transversality condition (14.14) is the same as (*) in Appendix A, from now we refer to (*).

[13] An equivalent approach to derivation of (14.15) can be based on applying the transversality condition (*) to the general solution formula for linear inhomogeneous first-order differential equations. Indeed, the first-order condition (14.11) provides such a differential equation in q_t.

Rearranging (14.11) and multiplying through by the integrating factor $e^{-\int_0^t r_s ds}$, we have

$$(r_t q_t - \dot{q}_t)e^{-\int_0^t r_s ds} = (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds}.$$

In Lemma A1, let $f(t) = q_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$. Then

$$\int_0^T (r_t q_t - \dot{q}_t)e^{-\int_0^t r_s ds}dt = q_0 - q_T e^{-\int_0^T r_s ds} = \int_0^T (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds}dt.$$

Rearranging and letting $T \to \infty$, we see that

$$q_0 = \int_0^\infty (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds}dt + \lim_{T\to\infty} q_T e^{-\int_0^T r_s ds}. \tag{14.45}$$

If, contrary to (*), $\lim_{T\to\infty} q_T e^{-\int_0^T r_s ds} > 0$ along the optimal path, then (14.45) shows that the firm is over-investing. By reducing initial investment by one unit, the firm would save approximately $1 + G_I(I_0, K_0) = q_0$, by (14.10), which would be more than the present value of the stream of potential net gains coming from this marginal unit of installed capital (the first term on the right-hand side of (14.45)).

Suppose instead that $\lim_{T\to\infty} q_T e^{-\int_0^T r_s ds} < 0$. Then, by a symmetric argument, the firm has under-invested initially.

Necessity of (14.12) In cases where along an optimal path, K_t remains bounded from above for $t \to \infty$, the transversality condition (14.12) is implied by (*). In cases where along an optimal path, K_t is not bounded from above for $t \to \infty$, the transversality condition (14.12) is stronger than (*). A proof of the necessity of (14.12) in this case can be based on Weitzman (2003) and Long and Shimomura (2003).

C. On different specifications of the q-theory

The simple relationship we have found between I and q can easily be generalized to the case where the purchase price on the investment good, p_{It}, is allowed to differ from 1 (its value above) and the capital installation cost is $p_{It}G(I_t, K_t)$. In this case it is convenient to replace q in the Hamiltonian function by, say, λ. Then the first-order condition (14.10) becomes $p_{It} + p_{It}G_I(I_t, K_t) = \lambda_t$, implying

$$G_I(I_t, K_t) = \frac{\lambda_t}{p_{It}} - 1,$$

and we can proceed, defining as before q_t by $q_t \equiv \lambda_t / p_{It}$.

Sometimes in the literature installation costs, J, appear in a slightly different form compared to the above exposition. But applied to a model with economic growth this will result in installation costs that rise faster than output and ultimately swallow the total produce.

Abel and Blanchard (1983), followed by Barro and Sala-i-Martin (2004, p. 152-160), introduce a function, ϕ, representing capital installation costs *per unit of investment* as a function of the investment-capital ratio. That is, total installation cost is $J = \phi(I/K)I$, where $\phi(0) = 0, \phi' > 0$. This implies that $J/K = \phi(I/K)(I/K)$. The right-hand side of this equation may be called $g(I/K)$, and then we are back at the formulation in Section 14.1. Indeed, defining $x \equiv I/K$, we have installation costs per unit of capital equal to $g(x) = \phi(x)x$, and assuming $\phi(0) = 0, \phi' > 0$, it holds that

$$\begin{aligned} g(x) &= 0 \text{ for } x = 0, \; g(x) > 0 \text{ for } x \neq 0, \\ g'(x) &= \phi(x) + x\phi'(x) \gtreqless 0 \text{ for } x \gtreqless 0, \text{ respectively, and} \\ g''(x) &= 2\phi'(x) + x\phi''(x). \end{aligned}$$

Now, $g''(x)$ must be positive for the theory to work. But the assumptions $\phi(0) = 0, \phi' > 0$, and $\phi'' \geq 0$, imposed in p. 153 and again in p. 154 in Barro and Sala-i-Martin (2004), are *not* sufficient for this (since $x < 0$ is possible). Since in macroeconomics $x < 0$ is seldom, this is only a minor point, of course. Yet, from a formal point of view the $g(\cdot)$ formulation may seem preferable to the $\phi(\cdot)$ formulation.

It is sometimes convenient to let the capital installation cost $G(I, K)$ appear, not as a reduction in output, but as a reduction in capital formation so that

$$\dot{K} = I - \delta K - G(I, K). \tag{14.46}$$

This approach is used in Hayashi (1982) and Heijdra and Ploeg (2002, p. 573 ff.). For example, Heijdra and Ploeg write the rate of capital accumulation as $\dot{K}/K = \varphi(I/K) - \delta$, where the "capital installation function" $\varphi(I/K)$ can be interpreted as $\varphi(I/K) \equiv [I - G(I, K)]/K = I/K - g(I/K)$; the latter equality comes from assuming G is homogeneous of degree 1. In one-sector models, as we usually consider in this text, this changes nothing of importance. In more general models this installation function approach may have some analytical advantages; what gives the best fit empirically is an open question. In our housing market model in the next chapter we apply a specification analogue to (14.46), interpreting $\dot{K}$ as the number of new houses per time unit.

Finally, some analysts assume that installation costs are a strictly convex function of *net* investment, $I - \delta K$, not gross investment, I. This agrees well with intuition if mere replacement investment occurs in a smooth way not involving new technology, work interruption, and reorganization. To the extent capital investment involves indivisibilities and embodies new technology, it may seem more plausible to specify the installation costs as a convex function of gross investment.

D. Proof of Hayashi's theorem

For convenience we repeat:

THEOREM (Hayashi) Assume the firm is a price taker, that the production function F is jointly concave in (K, L), and that the installation cost function G is jointly convex in (I, K). Then, along the optimal path we have:

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in (K, L) and/or G is strictly convex in (I, K).

Proof. The value of the firm as seen from time t is

$$V_t = \int_t^\infty (F(K_\tau, L_\tau) - G(I_\tau, K_\tau) - w_\tau L_\tau - I_\tau)e^{-\int_t^\tau r_s ds} d\tau. \tag{14.47}$$

We introduce the functions

$$\begin{aligned} A &= A(K, L) \equiv F(K, L) - F_K(K, L)K - F_L(K, L)L, & (14.48)\\ B &= B(I, K) \equiv G_I(I, K)I + G_K(I, K)K - G(I, K). & (14.49) \end{aligned}$$

Then the cash-flow of the firm at time τ can be written

$$\begin{aligned} R_\tau &= F(K_\tau, L_\tau) - F_{L\tau}L_\tau - G(I_\tau, K_\tau) - I_\tau \\ &= A(K_\tau, L_\tau) + F_{K\tau}K_\tau + B(I_\tau, K_\tau) - G_{I\tau}I_\tau - G_{K\tau}K_\tau - I_\tau, \end{aligned}$$

where we have used first $F_{L_\tau} = w$ and then the definitions of A and B above. Consequently, when moving along the optimal path,

$$\begin{aligned} V_t &= V^*(K_t, t) = \int_t^\infty (A(K_\tau, L_\tau) + B(I_\tau, K_\tau))\, e^{-\int_t^\tau r_s ds} d\tau & (14.50)\\ &\quad + \int_t^\infty [(F_{K\tau} - G_{K\tau})K_\tau - (1 + G_{I\tau})I_\tau]e^{-\int_t^\tau r_s ds} d\tau \\ &= \int_t^\infty (A(K_\tau, L_\tau) + B(I_\tau, K_\tau))e^{-\int_t^\tau r_s ds} d\tau + q_t K_t, \end{aligned}$$

cf. Lemma D1 below. Isolating q_t, it follows that

$$q_t^m \equiv q_t = \frac{V_t}{K_t} - \frac{1}{K_t}\int_t^\infty [A(K_\tau, L_\tau) + B(I_\tau, K_\tau)]e^{-\int_t^\tau r_s ds} d\tau, \tag{14.51}$$

when moving along the optimal path.

Since F is concave and $F(0,0)=0$, we have for all K and L, $A(K,L)\geq 0$ with equality sign, if and only if F is homogeneous of degree one. Similarly, since G is convex and $G(0,0)=0$, we have for all I and K, $B(I,K)\geq 0$ with equality sign, if and only if G is homogeneous of degree one. Now the conclusions (i) and (ii) follow from (14.51) and the definition of q^a in (14.27). □

LEMMA D1 The last integral on the right-hand side of (14.50) equals q_tK_t, when investment follows the optimal path.

Proof. We want to characterize a given optimal path $(K_\tau, I_\tau, L_\tau)_{\tau=t}^{\infty}$. Keeping t fixed and using z as our varying time variable, we have

$$(F_{Kz}-G_{Kz})K_z-(1+G_{Iz})I_z=[(r_z+\delta)q_z-\dot{q}_z]K_z-(1+G_{Iz})I_z$$
$$=[(r_z+\delta)q_z-\dot{q}_z]K_z-q_z(\dot{K}_z+\delta K_z)=r_zq_zK_z-(\dot{q}_zK_z+q_z\dot{K}_z)=r_zu_z-\dot{u}_z,$$

where we have used (14.11), (14.10), (14.6), and the definition $u_z\equiv q_zK_z$. We look at this as a differential equation: $\dot{u}_z-r_zu_z=\varphi_z$, where $\varphi_z\equiv-[(F_{Kz}-G_{Kz})K_z-(1+G_{Iz})I_z]$ is considered as some given function of z. The solution of this linear differential equation is

$$u_z=u_te^{\int_t^z r_s ds}+\int_t^z\varphi_\tau e^{\int_\tau^z r_s ds}d\tau,$$

implying, by multiplying through by $e^{-\int_t^z r_s ds}$, reordering, and inserting the definitions of u and φ,

$$\begin{aligned}&\int_t^z[(F_{K\tau}-G_{K\tau})K_\tau-(1+G_{I\tau})I_\tau]e^{-\int_t^\tau r_s ds}d\tau\\ =\;&q_tK_t-q_zK_ze^{-\int_t^z r_s ds}\to q_tK_t\quad\text{for}\quad z\to\infty,\end{aligned}$$

from the transversality condition (14.12) with t replaced by z and 0 replaced by t. □

A different – and perhaps more illuminating – way of understanding (i) in Hayashi's theorem is the following.

Suppose F and G are homogeneous of degree one. Then $A=B=0$, $G_II+G_KK=G=g(I/K)K$, and $F_K=f'(k)$, where f is the production function in intensive form. Consider an optimal path $(K_\tau, I_\tau, L_\tau)_{\tau=t}^{\infty}$ and let $k_\tau\equiv K_\tau/L_\tau$ and $x_\tau\equiv I_\tau/K_\tau$ along this path which we now want to characterize. As the path is assumed optimal, from (14.47) follows

$$V_t=V^*(K_t,t)=\int_t^\infty[f'(k_\tau)-g(x_\tau)-x_\tau]K_\tau e^{-\int_t^\tau r_s ds}d\tau. \tag{14.52}$$

From $\dot{K}_t = (x_t - \delta)K_t$ follows $K_\tau = K_t e^{-\int_t^\tau (x_s - \delta)ds}$. Substituting this into (14.52) yields

$$V^*(K_t, t) = K_t \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau] e^{-\int_t^\tau (r_s - x_s + \delta)ds} d\tau.$$

In view of (14.24), with t replaced by τ, the optimal investment *ratio* x_τ depends, for all τ, only on q_τ, not on K_τ, hence not on K_t. Therefore,

$$\partial V^*/\partial K_t = \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau] e^{-\int_t^\tau (r_s - x_s + \delta)ds} d\tau = V_t/K_t.$$

Hence, from (14.28) and (14.27), we conclude $q_t^m = q_t^a$.

Remark. We have assumed throughout that G is strictly convex in I. This does not imply that G is jointly strictly convex in (I, K). For example, the function $G(I, K) = I^2/K$ is strictly convex in I (since $G_{II} = 2/K > 0$). But at the same time this function has $B(I, K) = 0$ and is therefore homogeneous of degree one. Hence, it is not jointly strictly convex in (I, K).

E. The slope of the $\dot{q} = 0$ locus in the SOE case

First, we shall determine the sign of the slope of the $\dot{q} = 0$ locus in the case $g + n = 0$, considered in Fig. 14.4. Taking the total differential in (14.33) w.r.t. K and q gives

$$\begin{aligned} 0 &= -F_{KK}(K, \bar{L})dK + \{r + \delta + g'(m(q))m'(q) - [m(q) + (q-1)m'(q)]\} dq \\ &= -F_{KK}(K, \bar{L})dK + [r + \delta - m(q)] dq, \end{aligned}$$

since $g'(m(q)) = q - 1$, by (14.23) and (14.24). Therefore

$$\frac{dq}{dK}_{|\dot{q}=0} = \frac{F_{KK}(K, \bar{L})}{r + \delta - m(q)} \quad \text{for } r + \delta \neq m(q).$$

From this it is not possible to sign dq/dK at all points along the $\dot{q} = 0$ locus. But in a neighborhood of the steady state we have $m(q) \approx \delta$, hence $r + \delta - m(q) \approx r > 0$. And since $F_{KK} < 0$, this implies that at least in a neighborhood of E in Fig. 14.4 the $\dot{q} = 0$ locus is negatively sloped.

Second, consider the case $g + n > 0$, illustrated in Fig. 14.6. Here we get in a similar way

$$\frac{dq}{d\tilde{k}}_{|\dot{q}=0} = \frac{f''(\tilde{k}^*)}{r + \delta - m(q)} \quad \text{for } r + \delta \neq m(q).$$

From this it is not possible to sign $dq/d\tilde{k}$ at all points along the $\dot{q} = 0$ locus. But in a small neighborhood of the steady state we have $m(q) \approx \delta + \gamma + n$, hence $r + \delta - m(q) \approx r - \gamma - n$. Since $f'' < 0$, then, at least in a small neighborhood of E in Fig. 14.6, the $\dot{q} = 0$ locus is negatively sloped, when $r > \gamma + n$.

F. The divergent paths

Text not yet available.

14.7 Exercises

14.1 *(induced sluggish capital adjustment).* Consider a firm with capital installation costs $J = G(I, K)$, satisfying

$$G(0, K) = 0,\ G_I(0, K) = 0,\ G_{II}(I, K) > 0, \quad \text{and} \quad G_K(I, K) \leq 0.$$

a) Can we from this conclude anything as to strict concavity or strict convexity of the function G? If yes, with respect to what argument or arguments?

b) For two values of K, $\underline{K}$ and $\bar{K}$, illustrate graphically the capital installation costs J in the (I, J) plane. Comment.

c) By drawing a few straight line segments in the diagram, illustrate that $G(\frac{1}{2}I, \bar{K})2 < G(I, \bar{K})$ for any given $I > 0$.

14.2 (see end of Section 14.3)