Chapter 10

The basic representative agent model: Ramsey

As early as 1928 a sophisticated model of a society's optimal saving was published by the British mathematician and economist Frank Ramsey (1903-1930). Ramsey's contribution was mathematically demanding and did not experience much response at the time. Three decades had to pass until his contribution was taken up seriously (Samuelson and Solow, 1956). His model was merged with the growth model by Solow (1956) and became a cornerstone in neoclassical growth theory from the mid 1960s. The version of the model which we present below was completed by the work of Cass (1965) and Koopmans (1965). Hence the model is also known as the *Ramsey-Cass-Koopmans model*.

The model is one of the basic workhorse models in macroeconomics. As we conclude at the end of the chapter, the model can be seen as placed at one end of a line segment. At the other end appears another basic workhorse model, namely Diamond's overlapping generations model considered in chapters 3 and 4. While in the Diamond model there is an *unbounded* number of households (since in every new period a new generation enters the economy) and these have a *finite* time horizon, in the Ramsey model there is a *finite* number of households with an *unbounded* time horizon. Moreover, in the standard Ramsey model households are completely alike. The model is the main example of a *representative agent* model. In contrast, the Diamond model has heterogeneous agents, young versus old, interacting in every period. There are important economic questions where these differences in the setup lead to salient differences in the answers.

The purpose of this chapter is to describe and analyze the continuous-time version of the Ramsey framework. In the main sections we consider the case of a perfectly competitive market economy. In this context we shall see, for example, that the Solow growth model can be interpreted as a special case of the Ramsey model. toward the end of the chapter we consider the Ramsey framework in a setting with an "all-knowing and all-powerful" social planner.

10.1 Preliminaries

We consider a closed economy. Time is continuous. We assume households own the capital goods and hire them out to firms at a market *rental rate*, $\hat{r}$. This is just to have something concrete in mind. If instead the capital goods were owned by the firms using them in production, and the capital investment by these firms were financed by issuing shares and bonds, then the conclusions would remain the same as long as we ignore uncertainty.

Although time is considered continuous, to save notation, we shall write the time-dependent variables as w_t, $\hat{r}_t$, etc. instead of $w(t)$, $\hat{r}(t)$, etc. In every short time interval $(t, t+\Delta t)$, the individual firm employs labor at the market wage w_t and rents capital goods at the rental rate $\hat{r}_t$. The combination of labor and capital produces the homogeneous output good. This good can be used for consumption as well as investment. So in every short time interval there are at least three active markets, one for the "all-purpose", homogeneous output good, one for labor, and one for capital services (the rental market for capital goods). It may be convenient to imagine that there is also a perfect loan market. As all households are alike, however, the loan market will not be active in general equilibrium.

There is perfect competition in all markets, that is, households and firms are price takers. Any need for means of payment − money − is abstracted away. Prices are measured in units of the homogeneous output good.

There are no stochastic elements in the model. We assume households understand exactly how the economy works and can predict the future path of wages and interest rates. In other words, we assume "rational expectations". In our non-stochastic setting this amounts to *perfect foresight*. The results that emerge from the model are thereby the outcome of economic mechanisms in isolation from expectational errors.

Uncertainty being absent, rates of return on alternative assets are in equilibrium the same. In spite of the not active loan market, it is usual to speak of this common rate of return as the *real interest rate* of the economy. Denoting this rate r_t, for a given rental rate of capital, $\hat{r}_t$, we have

$$r_t = \frac{\hat{r}_t K_t - \delta K_t}{K_t} = \hat{r}_t - \delta, \tag{10.1}$$

where the right-hand side is the rate of return on holding K_t capital goods, δ (≥ 0) being a constant rate of capital depreciation. This relationship may be considered a *no-arbitrage condition* between investing in the loan market and in capital goods.

We describe, first, the households' behavior and next the firms' behavior. Thereafter the interaction between households and firms in general equilibrium and the resulting dynamics will be analyzed.

10.2 The agents

10.2.1 Households

There is a fixed number, N, of identical households with an infinite time horizon. This feature makes aggregation very simple: we just have to multiply the behavior of a single household with the number of households (for simplicity we later normalize N to equal 1). Every household has L_t (adult) members; L_t changes over time at a constant rate, n :

$$L_t = L_0 e^{nt}, \qquad L_0 > 0. \tag{10.2}$$

Indivisibility is ignored.

Each household member supplies inelastically one unit of labor per time unit. Equation (10.2) therefore describes the growth of both the population and the labor force. Since there is only one consumption good, the only decision problem is how to distribute current income between consumption and saving.

Intertemporal utility function

The household's preferences can be represented by an additive intertemporal utility function with a constant rate of time preference, ρ. Seen from time 0, the intertemporal utility function is

$$U_0 = \int_0^\infty u(c_t) L_t e^{-\rho t} dt,$$

where $c_t \equiv C_t/L_t$ is consumption per family member. The instantaneous utility function, $u(c)$, has $u'(c) > 0$ and $u''(c) < 0$, i.e., positive but diminishing marginal utility of consumption. The utility contribution from consumption per family member is weighted by the number of family members, L_t.

The household is seen as an infinitely-lived family, a family dynasty. The current members of the dynasty act in unity and are concerned about the utility from own consumption as well as the utility of the future generations within the dynasty.[1] Births (into adult life) do not amount to emergence of *new* economic agents with independent interests. Births and population growth are seen as just an expansion of the size of the already existing families. In contrast, in the Diamond OLG model births imply entrance of new economic decision makers whose preferences no-one cared about in advance.

In view of (10.2), U_0 can be written as

$$U_0 = \int_0^\infty u(c_t)e^{-(\rho-n)t}dt, \tag{10.3}$$

where the inconsequential positive factor L_0 has been eliminated. Here $\rho - n$ is known as the *effective* rate of time preference while ρ is the *pure* rate of time preference. We later introduce a restriction on $\rho - n$ to ensure boundedness from above of the utility integral in general equilibrium.

The household chooses a consumption-saving plan which maximizes U_0 subject to its budget constraint. Let $A_t \equiv a_t L_t$ be the household's (net) financial wealth in real terms at time t. We have

$$\dot{A}_t \equiv \frac{dA_t}{dt} = r_t A_t + w_t L_t - c_t L_t, \qquad A_0 \text{ given.} \tag{10.4}$$

This equation is a book-keeping relation telling how financial wealth or debt ($-A$) changes over time depending on how consumption relates to current income. The equation merely says that the increase in financial wealth per time unit equals saving which equals income minus consumption. Income is the sum of the net return on financial wealth, $r_t A_t$, and labor income, $w_t L_t$, where w_t is the real wage.[2] Saving can be negative. In that case the household dissaves and does so simply by selling a part of its stock of capital goods or by taking loans in the loan market. The market prices, w_t and r_t, faced by the household are assumed to be piecewise continuous functions of time.

When the dynamic budget identity (10.4) is combined with a requirement of solvency, we have a budget *constraint*. Given the assumed perfect loan market, the relevant solvency requirement is the No-Ponzi-Game condition (NPG for short):

$$\lim_{t\to\infty} A_t e^{-\int_0^t r_s ds} \geq 0. \tag{10.5}$$

This condition says that financial wealth far out in the future cannot have a negative present value. That is, in the long run, debt is at most allowed to rise

[1] The discrete-time Barro model of Chapter 7 articulated such an altruistic bequest motive. In that chapter we also discussed some of the conceptual difficulties associated with the dynasty setup.

[2] Since the technology exhibits constant returns to scale, in competitive equilibrium the firms make no (pure) profits to pay out to their owners.

at a rate *less* than the real interest rate r. The NPG condition thus precludes permanent financing of the interest payments by new loans.[3]

The decision problem is: choose a plan $(c_t)_{t=0}^{\infty}$ so as to maximize U_0 subject to non-negativity of the control variable, c, and the constraints (10.4) and (10.5). The problem is a slight generalization of the problem studied in Section 9.4 of the previous chapter.

To solve the problem we shall apply the Maximum Principle. This method can be applied directly to the problem as stated above or to an equivalent problem with constraints expressed in per capita terms. Let us follow the latter approach. From the definition $a_t \equiv A_t/L_t$ we get by differentiation w.r.t. t

$$\dot{a}_t = \frac{L_t \dot{A}_t - A_t \dot{L}_t}{L_t^2} = \frac{\dot{A}_t}{L_t} - a_t n.$$

Substitution of (10.4) gives the dynamic budget identity in per capita terms:

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given.} \tag{10.6}$$

By inserting $A_t \equiv a_t L_t = a_t L_0 e^{nt}$, the NPG condition (10.5) can be rewritten

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0, \tag{10.7}$$

where the unimportant factor L_0 has been eliminated.

We see that in both (10.6) and (10.7) a kind of corrected interest rate appears, namely the interest rate, r, minus the family size growth rate, n. Although deferring consumption gives a real interest rate of r, this return is diluted on a per capita basis because it will have to be shared with more members of the family when $n > 0$. In the form (10.7) the NPG condition requires that per capita debt, if any, in the long run at most grows at a rate *less* than $r - n$, assuming the interest rate is a constant, r.

Solving the consumption-saving problem

The decision problem is now: choose $(c_t)_{t=\infty}^{0}$ so as to a maximize U_0 subject to the constraints: $c_t \geq 0$, (10.6), and (10.7). To solve the problem we apply the

[3]From the previous chapter we know that the NPG condition, in combination with (10.4), is equivalent to an ordinary *intertemporal* budget constraint which says that the present value of the planned consumption path cannot exceed initial total wealth, i.e., the sum of the initial financial wealth and the present value of expected future labor income.

Violating the NPG condition means running a "Ponzi game", that is, trying to make a fortune through the chain-letter principle where old investors are payed off with money from the new investors.

Maximum Principle. So we follow the same solution procedure as in the alike problem (apart from $n = 0$) of Section 9.4 of the previous chapter:

1) Set up the current-value Hamiltonian

$$H(a, c, \lambda, t) = u(c) + \lambda \left[(r - n)\, a + w - c \right],$$

where λ is the *adjoint variable* associated with the differential equation (10.6).

2) Differentiate H partially w.r.t. the control variable, c, and put the result equal to zero:

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0. \tag{10.8}$$

3) Differentiate H partially w.r.t. the state variable, a, and put the result equal to minus the time derivative of λ plus the effective discount rate (appearing in the integrand of the criterion function) multiplied by λ:

$$\frac{\partial H}{\partial a} = \lambda(r - n) = -\dot{\lambda} + (\rho - n)\lambda. \tag{10.9}$$

4) Apply the Maximum Principle: an interior optimal path, $(a_t, c_t)_{t=0}^{\infty}$, will satisfy that there exists a continuous function $\lambda = \lambda_t$ such that for all $t \geq 0$, (10.8) and (10.9) hold along the path, and the transversality condition,

$$\lim_{t \to \infty} a_t \lambda_t e^{-(\rho - n)t} = 0, \tag{10.10}$$

is satisfied.[4]

The interpretation of these optimality conditions is as follows. The condition (10.8) can be considered a $MC = MB$ condition (in utility terms). It illustrates together with (10.9) that the adjoint variable, λ, constitutes the shadow price, measured in current utility, of per head financial wealth along the optimal path. In the differential equation (10.9) λn cancels out, and rearranging (10.9) gives

$$\frac{r\lambda + \dot{\lambda}}{\lambda} = \rho.$$

This can be interpreted as a no-arbitrage condition. The left-hand side gives the *actual* rate of return, measured in utility units, on the marginal unit of saving: $r\lambda$ can be seen as a dividend and $\dot{\lambda}$ as a capital gain. The right-hand side is the *required* rate of return in utility units, ρ. The household is willing to save the

[4] That in the present problem, optimality does indeed require the "standard" condition (10.10) satisfied is true (as shown in Chapter 9.4). It is not a *general* result contained in the Maximum Principle.

marginal unit of income only up to the point where the actual return on saving equals the required return.

The transversality condition (10.10) says that optimality requires that the present shadow value of per capita financial wealth goes to zero for $t \to \infty$. Combined with (10.8), the condition can be written

$$\lim_{t\to\infty} a_t u'(c_t) e^{-(\rho-n)t} = 0. \tag{10.11}$$

This requirement is not surprising if we compare with the alternative case where $\lim_{t\to\infty} a_t u'(c_t) e^{-(\rho-n)t} > 0$. In this case there would be over-saving; U_0 could be increased by reducing the long-run a_t through consuming more and thereby saving less. The opposite inequality, $\lim_{t\to\infty} a_t u'(c_t) e^{-(\rho-n)t} < 0$, will reflect over-consumption and not even satisfy the NPG condition in view of Proposition 2 of the previous chapter. In fact, from that proposition we know that the transversality condition (10.10) is equivalent to the NPG condition (10.7) being satisfied with strict equality, i.e.,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0. \tag{10.12}$$

We should recall that the Maximum Principle gives only *necessary* conditions for an optimal plan. But since the Hamiltonian is jointly concave in (a, c) for every t, the necessary conditions are also *sufficient*, by Mangasarian's sufficiency theorem (Math Tools).

The first-order conditions (10.8) and (10.9) give the Keynes-Ramsey rule:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(r_t - \rho), \tag{10.13}$$

where $\theta(c_t)$ is the (absolute) elasticity of marginal utility,

$$\theta(c_t) \equiv -\frac{c_t}{u'(c_t)} u''(c_t) > 0. \tag{10.14}$$

As we know from previous chapters, this elasticity measures the consumer's wish to smooth consumption over time. The inverse of $\theta(c_t)$ is the elasticity of intertemporal substitution in consumption. It measures the strength of the willingness to vary consumption over time in response to a change in the interest rate.

Note that the population growth rate, n, does not appear in the Keynes-Ramsey rule. Going from $n = 0$ to $n > 0$ implies that r_t is replaced by $r_t - n$ in the dynamic budget identity (10.6) and ρ is replaced by $\rho - n$ in the criterion function. Hence n cancels out in the Keynes-Ramsey rule. Yet n appears in the transversality condition and is thereby a co-determinant of the *level* of consumption for given wealth, cf. (10.18) below.

CRRA utility

In order that the model can accommodate Kaldor's stylized facts, it should be capable of generating a balanced growth path. When the population grows at the same constant rate as the labor force, here n, by definition balanced growth requires that per capita output, per capita capital, and per capita consumption grow at constant rates. At the same time another of Kaldor's stylized facts is that the general rate of return in the economy tends to be constant. But (10.13) shows that having a constant per capita consumption growth rate at the same time as r is constant, is only possible if the elasticity of marginal utility does *not* vary with c. Hence, it makes sense to assume that the right-hand-side of (10.14) is a positive constant, θ. We thus assume that the instantaneous utility function is of CRRA form:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0; \tag{10.15}$$

where, for $\theta = 1$, the right-hand side should be interpreted as $\ln c$ as explained in Section 3.3 of Chapter 3.[5]

In later sections of this chapter we let the time horizon of the decision maker go to infinity. To ease convergence of an infinite sum of discounted utilities, it is an advantage not to have to bother with additive constants in the period utilities and therefore we write the CRRA function as $c^{1-\theta}/(1-\theta)$ instead of the form, $(c^{1-\theta}-1)/(1-\theta)$, introduced in Chapter 3. As implied by Box 9.1, the two forms represent the same preferences.

So our Keynes-Ramsey rule simplifies to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho). \tag{10.16}$$

The consumption function* The Keynes-Ramsey rule characterizes the optimal *rate of change* of consumption. The optimal initial *level* of consumption, c_0, will be the highest feasible c_0 which is compatible with both the Keynes-Ramsey rule and the NPG condition. And for this reason the choice of c_0 will exactly comply with the transversality condition (10.12). Although at this stage an explicit determination of c_0 is not necessary to pin down the equilibrium path of the economy (see below), we note in passing that c_0 can be found by the method described at the end of Chapter 9. Indeed, given the book-keeping relation (10.6), we know from Proposition 1 of that chapter that the transversality condition

[5] As mentioned in the previous chapter, in problems with infinite horizon it is an advantage not to have to bother with additive constants in the instantaneous utilities. Otherwise, convergence of the improper integral (10.3) may go by the board. Hence we write the CRRA function as in (10.15), without subtraction of the constant $1/1-\theta$).

(10.12) is equivalent to satisfying the intertemporal budget constraint with strict equality:

$$\int_0^\infty c_t e^{-\int_0^t (r_s - n)ds} dt = a_0 + h_0. \tag{10.17}$$

Solving the differential equation (10.16), we get $c_t = c_0 e^{\frac{1}{\theta}\int_0^t (r_s - \rho)ds}$ which we substitute for c_t in (10.17). Isolating c_0 now gives[6]

$$\begin{aligned} c_0 &= \beta_0(a_0 + h_0), \quad \text{where} \qquad (10.18) \\ \beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r_s - \rho}{\theta} + n)ds} dt}, \quad \text{and} \\ h_0 &= \int_0^\infty w_t e^{-\int_0^t (r_s - n)ds} dt. \end{aligned}$$

Initial consumption is thus proportional to total wealth. The factor of proportionality is β_0, also called the marginal (and average) propensity to consume out of wealth. We see that the entire expected future evolution of wages and interest rates affects c_0 through β_0. Moreover, β_0 is less, the greater is the population growth rate, n.[7] The explanation is that the effective utility discount rate, $\rho - n$, is less, the greater is n. The propensity to save is greater the more mouths to feed in the future. The initial saving level will be $r_0 a_0 + w_0 - c_0 = r_0 a_0 + w_0 - \beta_0(a_0 + h_0)$.

In case $r_t = r$ for all t and $w_t = w_0 e^{gt}$, where $g < r - n$, we get $\beta_0 = [(\theta - 1)r + \rho - \theta n]/\theta$ and $a_0 + h_0 = a_0 + w_0/(r - n - g)$.

In the Solow growth model the saving-income ratio is a parameter, a given constant. The Ramsey model endogenizes the saving-income ratio. Solow's parametric saving-income ratio is replaced by two "deeper" parameters, the rate of impatience, ρ, and the desire for consumption smoothing, θ. As we shall see, the resulting saving-income ratio will not generally be constant outside the steady state of the dynamic system implied by the Ramsey model. But first we need a description of production.

10.2.2 Firms

There is a large number of firms. They have the same neoclassical production function with CRS,

$$Y_t = F(K_t^d, T_t L_t^d) \tag{10.19}$$

where Y_t is supply of output, K_t^d is capital input, and L_t^d is labor input, all measured per time unit, at time t. The superscript d on the two inputs indicates

[6] These formulas can also be derived directly from Example 1 of Chapter 9.5 by replacing $r(\tau)$ and ρ by $r(\tau) - n$ and $\rho - n$, respectively. As to h_0, see the hint in Exercise 9.1.

[7] This holds also if $\theta = 1$, i.e., $u(c) = \ln c$, since in that case $\beta_0 = \rho - n$.

that these inputs are seen from the demand side. The factor T_t represents the economy-wide level of technology as of time t and is exogenous. We assume there is technological progress at a constant rate g (≥ 0) :

$$T_t = T_0 e^{gt}, \qquad T_0 > 0. \tag{10.20}$$

Thus the economy features Harrod-neutral technological progress, as is needed for compliance with Kaldor's stylized facts.

Necessary and sufficient conditions for the factor combination (K_t^d, L_t^d), where $K_t^d > 0$ and $L_t^d > 0$, to maximize profits under perfect competition are that

$$F_1(K_t^d, T_t L_t^d) = \hat{r}_t \equiv r_t + \delta, \tag{10.21}$$

$$F_2(K_t^d, T_t L_t^d) T_t = w_t, \tag{10.22}$$

$\hat{r}_t$ being the rental rate of capital, cf. (10.1).

10.3 General equilibrium and dynamics

We now consider the economy as a whole and thereby the interaction between households and firms in the various markets. For simplicity, we assume that the number of households, N, is the same as the number of firms. We normalize this common number to *one* so that $F(\cdot, \cdot)$ from now on is interpreted as the aggregate production function and C_t as aggregate consumption.

Factor markets

In the short term, i.e., for fixed t, the available quantities of labor, $L_t = L_0 e^{nt}$, and capital, K_t, are predetermined. The factor markets clear at all points in time, that is,

$$K_t^d = K_t, \quad \text{and} \quad L_t^d = L_t, \qquad \text{for all } t \geq 0. \tag{10.23}$$

It is the rental rate, $\hat{r}_t$, and the wage rate, w_t, which adjust (immediately) so that this is achieved for every t. Aggregate output can be written

$$Y_t = F(K_t, T_t L_t) = T_t L_t F(\tilde{k}_t, 1) \equiv T_t L_t f(\tilde{k}_t), \qquad f' > 0, f'' < 0, \tag{10.24}$$

where $\tilde{k}_t \equiv k_t / T_t \equiv K_t / (T_t L_t)$ is the effective capital-labor ratio, also sometimes just called the "capital intensity". Substituting (10.23) into (10.21) and (10.22), we find the equilibrium interest rate and wage rate:

$$r_t = \hat{r}_t - \delta = \frac{\partial (T_t L_t f(\tilde{k}_t))}{\partial K_t} - \delta = f'(\tilde{k}_t) - \delta, \tag{10.25}$$

$$w_t = \frac{\partial (T_t L_t f(\tilde{k}_t))}{\partial (T_t L_t)} T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] T_t \equiv \tilde{w}(\tilde{k}_t) T_t, \tag{10.26}$$

where $\tilde{k}_t$ is at any point in time predetermined and where in (10.25) we have used the no-arbitrage condition (10.1).

Capital accumulation

From now on we leave out the explicit dating of the variables when not needed for clarity. By national product accounting we have

$$\dot{K} = Y - C - \delta K. \tag{10.27}$$

Let us check whether we get the same result from the wealth accumulation equation of the household. Because physical capital is the only asset in the economy, aggregate financial wealth, A, at time t equals the total quantity of capital, K, at time t.[8] With S^N denoting aggregate net saving, we thus have

$$\begin{aligned}
\dot{K} &= \dot{A} = S^N = rK + wL - cL && \text{(by (10.4))} \\
&= (f'(\tilde{k}) - \delta)K + (f(\tilde{k}) - \tilde{k}f'(\tilde{k}))TL - cL && \text{(by (10.25) and (10.26))} \\
&= f(\tilde{k})TL - \delta K - cL && \text{(by rearranging and use of } K \equiv \tilde{k}TL) \\
&= F(K, TL) - \delta K - C = Y - C - \delta K && \text{(by } C \equiv cL).
\end{aligned}$$

Hence the book-keeping is in order (the national product account is consistent with the national income account).

We now face an important difference as compared with models where households have a finite horizon, such as the Diamond OLG model. Current consumption cannot be determined independently of the expected entire future evolution of the economy. Consumption and saving, as we saw in Section 10.2, depend on the expectations of the future path of wages and interest rates. And given the presumption of rational expectations, here in the form of perfect foresight, the households' expectations are identical to the prediction that can be calculated from the model. In this way there is mutual dependence between expectations and the level and evolution of consumption. We can determine the level of consumption only in the context of the overall dynamic analysis. In fact, the economic agents are in some sense in the same situation as the outside analyst. They, too, have to think through the entire dynamics of the economy, including the mutual dependency between expectations and actual evolution, in order to form their rational expectations.

The dynamic system

We get a concise picture of the dynamics by reducing the model to the minimum number of coupled differential equations. This minimum number is two. The key

[8] Whatever financial claims on each other the households might have, they net out for the household sector as a whole.

endogenous variables are $\tilde{k} \equiv K/(TL)$ and $\tilde{c} \equiv C/(TL) \equiv c/T$. Using the rule for the growth rate of a quotient, we get

$$\begin{aligned}
\frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{T}}{T} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - (g+n) \qquad \text{(from (10.2) and (10.20))} \\
&= \frac{F(K,TL) - C - \delta K}{K} - (g+n) \qquad \text{(from (10.27))} \\
&= \frac{f(\tilde{k}) - \tilde{c}}{\tilde{k}} - (\delta + g + n) \qquad \text{(from (10.24))}.
\end{aligned}$$

The associated differential equation for $\tilde{c}$ is obtained in a similar way:

$$\begin{aligned}
\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{T}}{T} = \frac{1}{\theta}(r_t - \rho) - g \quad \text{(from the Keynes-Ramsey rule)} \\
&= \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right] \qquad \text{(from (10.25))}.
\end{aligned}$$

We thus end up with the dynamic system

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}, \qquad \tilde{k}_0 > 0 \quad \text{given}, \tag{10.28}$$

$$\dot{\tilde{c}} = \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right]\tilde{c}. \tag{10.29}$$

There is no given initial value of c. Instead we have the transversality condition (10.12). Using $a_t = K_t/L_t \equiv \tilde{k}_t T_t = \tilde{k}_t T_0 e^{gt}$ and $r_t = f'(\tilde{k}_t) - \delta$, we see that (10.12) is equivalent to

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left(f'(\tilde{k}_s) - \delta - g - n\right) ds} = 0. \tag{10.30}$$

Phase diagram By a *phase diagram* for the dynamic system (10.28) – (10.29) is meant a graph in the $(\tilde{k}, \tilde{c})$ plane showing projections of the time paths, $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$, that are consistent with the system for alternative arbitrary initial points, $(\tilde{k}_0, \tilde{c}_0)$. The phase diagram is shown in Fig. 10.2 below.

Fig. 10.1 is an aid for the construction of the phase diagram in Fig. 10.2. The curve OEB in Fig. 10.2 represents the points in the $(\tilde{k}, \tilde{c})$ plane where $\dot{\tilde{k}} = 0$ according to the differential equation (10.28). Such a curve is called a *nullcline* for $\tilde{k}$. We see from (10.28) that

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + g + n)\tilde{k} \equiv \tilde{c}(\tilde{k}). \tag{10.31}$$

The value of $\tilde{c}(\tilde{k})$ for alternative values of $\tilde{k}$ can be read off in Fig. 10.1 as the vertical distance between the curve $\tilde{y} = f(\tilde{k})$ and the line $\tilde{y} = (\delta + g + n)\tilde{k}$ (to save space, the proportions are somewhat distorted).[9] The maximum value of $\tilde{c}(\tilde{k})$, if it exists, is reached at the point where the tangent to the OEB curve in Fig. 10.2 is horizontal, i.e., where $\tilde{c}'(\tilde{k}) = f'(\tilde{k}) - (\delta + g + n) = 0$ or $f'(\tilde{k}) - \delta = g + n$. The value of $\tilde{k}$ satisfying this is the golden-rule capital intensity, $\tilde{k}_{GR}$:

$$f'(\tilde{k}_{GR}) - \delta = g + n. \tag{10.32}$$

By (10.28) follows that $\partial\dot{\tilde{k}}/\partial\tilde{c} = -1$. For points above the $\dot{\tilde{k}} = 0$ locus we thus have $\dot{\tilde{k}} < 0$, whereas for points below the $\dot{\tilde{k}} = 0$ locus, $\dot{\tilde{k}} > 0$. The horizontal arrows in the figure indicate these directions of movement of $\tilde{k}$ in the different regions.

We see from (10.29) that

$$\dot{\tilde{c}} = 0 \text{ for } f'(\tilde{k}) = \delta + \rho + \theta g \qquad \text{or} \qquad \tilde{c} = 0. \tag{10.33}$$

Let $\tilde{k}^* > 0$ satisfy the equation $f'(\tilde{k}^*) - \delta = \rho + \theta g$. Then the vertical half-line $\tilde{k} = \tilde{k}^*$, $\tilde{c} \geq 0$, represents points where $\dot{\tilde{c}} = 0$, and so does the horizontal half-line $\tilde{c} = 0, \tilde{k} \geq 0$. These two half-lines thus make up *nullclines* for $\tilde{c}$ according to the differential equation (10.29).

By (10.28) follows that for $\tilde{c} > 0$, $\partial\dot{\tilde{c}}/\partial\tilde{k} = \theta^{-1}f''(\tilde{k})\tilde{c} < 0$. For points to the left of the $\tilde{k} = \tilde{k}^*$ line we thus have $\dot{\tilde{c}} > 0$. And for points to the right of the $\tilde{k} = \tilde{k}^*$ line we have $\dot{\tilde{c}} < 0$. The vertical arrows in Fig. 10.2 indicate these directions of movement of $\tilde{c}$ in the different regions. Four illustrative examples of solution curves (*I*, *II*, *III*, and *IV*) are drawn in the figure. Since our dynamic system is "autonomous", the direction of movement depends only on the initial position, not on time. Hence, generally in a phase diagram the time index on $\tilde{k}$ and $\tilde{c}$ is omitted.

[9] As the graph is drawn, $f(0) = 0$, i.e., capital is assumed essential. But none of the conclusions we are going to consider depends on this.

Steady state

The point E in Fig. 10.2 has coordinates ($\tilde{k}^*$, $\tilde{c}^*$) and represents the unique steady state.[10] From (10.33) and (10.31), respectively, follows that

$$\begin{aligned} f'(\tilde{k}^*) &= \delta + \rho + \theta g, \qquad \text{and} & (10.34) \\ \tilde{c}^* &= f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*. & (10.35) \end{aligned}$$

So, in steady state the real interest rate is

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g. \qquad (10.36)$$

The effective capital-labor ratio satisfying this equation is known as the *modified-golden-rule* capital intensity, $\tilde{k}_{MGR}$. The modified golden rule is the rule saying that for a representative agent economy to be in steady state, the capital intensity must be such that the net marginal productivity of capital equals the required rate of return, taking into account the pure rate of time preference, ρ, and the desire for consumption smoothing, θ.[11]

We show below that the steady state is, in a specific sense, asymptotically stable. First we have to make sure, however, that the steady state is consistent with general equilibrium. This consistency requires that the household's transversality condition (10.30) holds in the point E, where, for all $t \geq 0$, $\tilde{k}_t = \tilde{k}^*$ and $f'(\tilde{k}_t) - \delta = \rho + \theta g$. So the condition (10.30) becomes

$$\lim_{t\to\infty} \tilde{k}^* e^{-(\rho+\theta g-g-n)t} = 0. \qquad (10.37)$$

This is fulfilled if and only if $\rho + \theta g > g + n$, a condition equivalent to

$$\rho - n > (1 - \theta)g. \qquad \text{(A1)}$$

This "sufficient impatience" condition also ensures that the improper integral U_0 is bounded from above (see Appendix B). If $\theta \geq 1$, (A1) is fulfilled as soon as

[10] As (10.33) shows, if $\tilde{c}_t = 0$, then $\dot{\tilde{c}} = 0$. Therefore, mathematically, point B (if it exists) in Fig. 10.2 is also a stationary point of the dynamic system. And if $f(0) = 0$, then according to (10.29) and (10.31) also the point $(0, 0)$ in the figure is a stationary point. But these stationary points have zero consumption forever and are therefore not steady states of any *economic* system. That is, they are "trivial" steady states.

[11] The ρ of the Ramsey model corresponds to the intergenerational discount rate R of Barro's dynasty model in Chapter 7. In the discrete time Barro model we have $1+r^* = (1+R)(1+g)^\theta$, which, by taking logs on both sides and using first-order Taylor approximations of $\ln(1 + x)$ around $x = 0$ gives $r^* \approx \ln(1 + r^*) = \ln(1 + R) + \theta \ln(1 + g) \approx R + \theta g$. Recall, however, that in view of the considerable period length (about 25-30 years) of the Barro model, this approximation may not be good.

the effective utility discount rate, $\rho - n$, is positive. (A1) may even hold for a negative $\rho - n$ if not "too" negative. If $\theta < 1$, (A1) requires $\rho - n$ to be "sufficiently positive".

Since the parameter restriction (A1) can be written $\rho + \theta g > g + n$, it implies that the steady-state interest rate, r^*, given in (10.36), is higher than the "natural" growth rate, $g + n$. If this did not hold, the transversality condition (10.12) would fail at the steady-state point E. Indeed, along the steady-state path we have

$$a_t e^{-(r^*-n)t} = k_t e^{-(r^*-n)t} = k_0 e^{gt} e^{-(r^*-n)t} = k_0 e^{(g+n-r^*)t},$$

which would take the constant positive value k_0 for all $t \geq 0$ if $r^* = g + n$ and would go to ∞ for $t \to \infty$ if $r^* < g + n$. The individual households would thus be over-saving. Each household would in this situation alter its behavior and the steady state could not be an equilibrium path.

Another way of seeing that $r^* \leq g + n$ can not be an equilibrium in a Ramsey model is to recognize that this condition would make the infinitely-lived household's human wealth $= \infty$ because wage income, wL, would grow at a rate, $g + n$, at least as high as the real interest rate, r^*. This would motivate an immediate increase in consumption and so the considered steady-state path would again not be an equilibrium.

To have a model of interest, from now on we assume that the preference and technology parameters satisfy the inequality (A1). As an implication, the effective capital-labor ratio in steady state, $\tilde{k}^*$, is less than the golden-rule value $\tilde{k}_{GR}$. Indeed, $f'(\tilde{k}^*) - \delta = \rho + \theta g > g + n = f'(\tilde{k}_{GR}) - \delta$, so that $\tilde{k}^* < \tilde{k}_{GR}$, in view of $f'' < 0$.

So far we have only ensured that *if* the steady state, E, exists, it is consistent with general equilibrium. Existence of a steady state requires that the marginal productivity of capital is sufficiently sensitive to variation in the effective capital-labor ratio:

$$\lim_{\tilde{k} \to 0} f'(\tilde{k}) - \delta > \rho + \theta g > \lim_{\tilde{k} \to \infty} f'(\tilde{k}) - \delta. \tag{A2}$$

We could proceed with this assumption. To allow comparison of the steady state of the model with a golden rule allocation, we need that a golden rule allocation exists. This requires that $\lim_{\tilde{k} \to 0} f'(\tilde{k}) - \delta > g + n > \lim_{\tilde{k} \to \infty} f'(\tilde{k}) - \delta$. This together with both (A2) and (A1) gives the "synthesized" condition

$$\lim_{\tilde{k} \to 0} f'(\tilde{k}) - \delta > \rho + \theta g > g + n > \lim_{\tilde{k} \to \infty} f'(\tilde{k}) - \delta. \tag{A2'}$$

By continuity of f', these inequalities ensure the existence of both $\tilde{k}^*$ and $\tilde{k}_{GR}$ such that $0 < \tilde{k}^* < \tilde{k}_{GR}$.[12] As illustrated by Fig. 10.1, the inequalities also ensure

[12] The often presumed Inada conditions, $\lim_{\tilde{k} \to 0} f'(\tilde{k}) = \infty$ and $\lim_{\tilde{k} \to \infty} f'(\tilde{k}) = 0$, are stricter than both (A2) and (A2') and are not necessary.

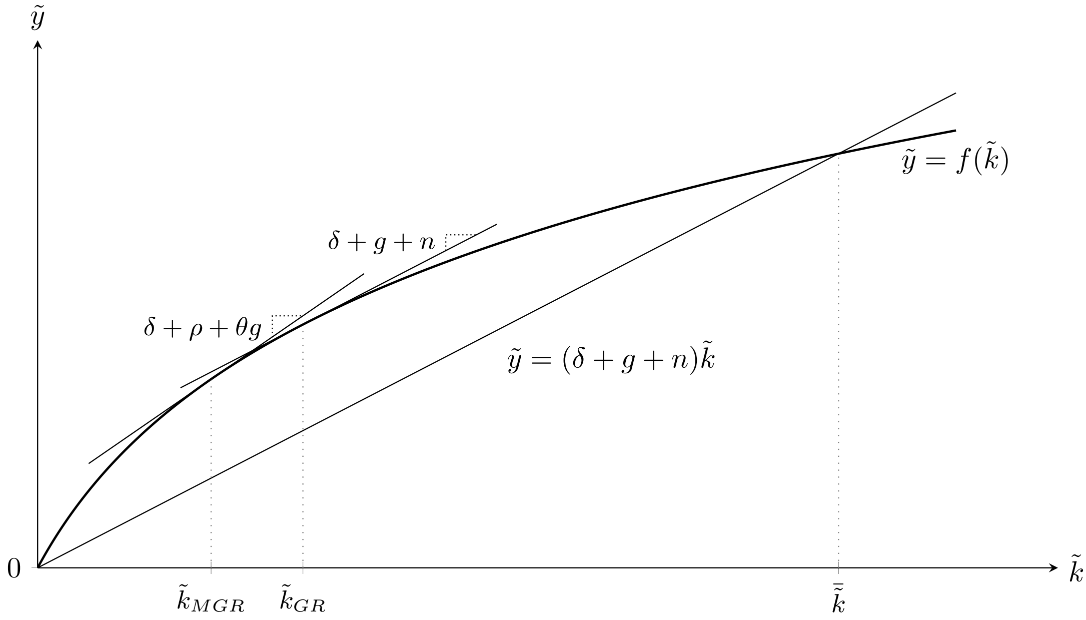

Figure 10.1: Building blocks for the phase diagram.

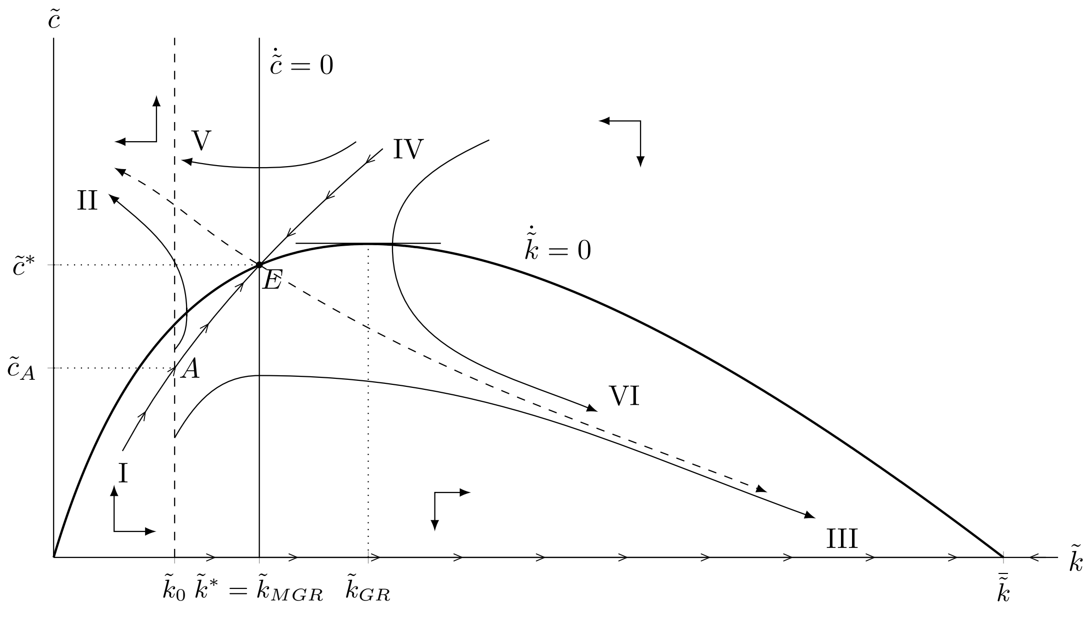

Figure 10.2: Phase diagram for the Ramsey model.

existence of a $\bar{\tilde{k}} > 0$ with the property that $f(\bar{\tilde{k}}) - (\delta + g + n)\tilde{k} = 0.$[13] Because $f'(\tilde{k}) > 0$ for all $\tilde{k} > 0$, it is implied by the assumption (A2') that $\delta + g + n > 0$. Even without deciding on the sign of n (a decreasing workforce should not be ruled out in our days), this inequality seems a plausible presumption.

Trajectories in the phase diagram

A first condition for a path $(\tilde{k}_t,\ \tilde{c}_t)$, with $\tilde{k}_t > 0$ and $\tilde{c}_t > 0$ for all $t \geq 0$, to be a solution to the model is that it satisfies the system of differential equations (10.28)-(10.29). Indeed, to be technically feasible, it must satisfy (10.28) and to comply with the Keynes-Ramsey rule, it must satisfy (10.29). Technical feasibility of the path also requires that the initial value for $\tilde{k}$ equals the historically given value $\tilde{k}_0 \equiv K_0/(T_0 L_0)$. In contrast, for $\tilde{c}$ we have no given initial value. This is because $\tilde{c}_0$ is a *jump variable,* also known as a *forward-looking variable.* These names are used for an endogenous variable which can immediately shift to another value if new information arrives so as to alter expectations about the future. We shall see that the terminal condition (10.30), reflecting the transversality condition of the households, makes up for this lack of an initial condition for c.

In Fig. 10.2 we have drawn some paths that are consistent with our dynamic system (10.28)-(10.29). We are especially interested in the paths which are consistent with the historically given $\tilde{k}_0$, that is, paths starting at some point on the stippled vertical line in the figure. If the economy started out with a "high" value of $\tilde{c}$, it would follow a curve like II in the figure. The low level of saving implies that the capital stock goes to zero in finite time (see Appendix C). If the economy starts out with a "low" level of $\tilde{c}$, it will follow a curve like III in the figure. The high level of saving implies that the effective capital-labor ratio converges toward $\bar{\tilde{k}}$ in the figure.

All in all this suggests the existence of an initial level of consumption somewhere in between, which results in a path like I. Indeed, since the curve II emerged with a high $\tilde{c}_0$, then by lowering this $\tilde{c}_0$ slightly, a path will emerge in which the maximal value of $\tilde{k}$ on the $\dot{\tilde{k}} = 0$ locus is greater than curve II's maximal $\tilde{k}$ value.[14] We continue lowering $\tilde{c}_0$ until the path's maximal $\tilde{k}$ value is exactly equal to $\tilde{k}^*$, where the the path ends. The path which emerges from this, namely the path I, starting at the point A, is special in that it converges toward

[13] We claim that $\bar{\tilde{k}} > \tilde{k}_{GR}$ *must* hold. Indeed, this inequality follows from $f'(\tilde{k}_{GR}) = \delta + n + g \equiv f(\bar{\tilde{k}})/\bar{\tilde{k}} > f'(\bar{\tilde{k}})$, the latter inequality being due to $f'' < 0$ and $f(0) \geq 0$ (consider the graph of $f(\tilde{k})$).

[14] As an implication of the uniqueness theorem for differential equations (see Math Tools), two solution paths in the phase plane cannot intersect.

the steady-state point E. No other path starting at the stippled line, $\tilde{k} = \tilde{k}_0$, has this property. Paths starting above A do not, as we just saw. Neither do paths starting below A, like path III. Either this path never reaches the consumption level $\tilde{c}_A$ in which case it can not converge to E, of course. Or, after a while its consumption level reaches $\tilde{c}_A$, but at the same time it must have $\tilde{k} > \tilde{k}_0$. From then on, as long as $\tilde{k} \leq \tilde{k}^*$, for every $\tilde{c}$-value that path III has in common with path I, path III has a higher $\dot{\tilde{k}}$ and a lower $\dot{\tilde{c}}$ than path I (use (10.28) and (10.29)). Hence, path III diverges from point E.

Had we considered a value of $\tilde{k}_0 > \tilde{k}^*$, there would similarly be a unique value of $\tilde{c}_0$ such that the path starting from $(\tilde{k}_0, \tilde{c}_0)$ would converge to E (see path IV in Fig. 10.2).

The point E is a *saddle point*. By this is meant a steady state with the following property: there exists exactly two paths in the phase plane, one from each side of $\tilde{k}^*$, that converge toward the steady-state point. All other paths (at least if starting in a neighborhood of the steady state) move away from the steady state and asymptotically approach one of the two diverging paths, the stippled North-West and South-East curves in Fig. 10.2.[15] The two converging paths are called *saddle paths*.[16] In combination they make up what is known as the *stable branch* (or *stable arm*). The stippled diverging paths in Fig. 10.2, together, make up the *unstable branch* (or *unstable arm*).

The equilibrium path

A solution to the model is a path which is technically feasible and satisfies a set of equilibrium conditions. In analogy with the definition in discrete time (see Chapter 3) a path $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$ is called a *technically feasible path* if (i) the path has $\tilde{k}_t \geq 0$ and $\tilde{c}_t \geq 0$ for all $t \geq 0$; (ii) it satisfies the accounting equation (10.28); and (iii) it starts out, at $t = 0$, with the historically given initial effective capital-labor ratio. An *equilibrium path* with perfect foresight is then a technically feasible path $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$ with the properties that the path (a) is consistent with firms' profit maximization and households' optimization given their expectations and budget constraints; (b) is consistent with market clearing for all $t \geq 0$; and (c) has the property that the evolution of the pair (w_t, r_t), where $w_t = \tilde{w}(\tilde{k}_t)T_t$ and $r_t = f'(\tilde{k}_t) - \delta$, is as expected by the households. Among other things these conditions require the (transformed) Keynes-Ramsey rule, (10.29), and the transversality condition, (10.30), to hold for all $t \geq 0$.

Consider the case illustrated in Fig. 10.2, where $0 < \tilde{k}_0 < \tilde{k}^*$. The path which

[15] The algebraic definition of a saddle point, in terms of eigenvalues, is given in Appendix A.

[16] If $\lim_{\tilde{k} \to 0} f(\tilde{k}) = 0$, then the saddle path on the left-hand side of the steady state in Fig. 10.2 will start out infinitely close to the origin, see Appendix A.

starts at point A and follows the saddle path toward the steady state is an equilibrium path because, by construction, it is technically feasible and in addition has the required properties, (a), (b), and (c). More intuitively: if the households expect an evolution of w_t and r_t corresponding to this path (that is, expect a corresponding underlying movement of $\tilde{k}_t$, which we know unambiguously determines r_t and w_t), then these expectations will induce a behavior the aggregate result of which is an actual path for $(\tilde{k}_t, \tilde{c}_t)$ that confirms the expectations. And along this path the households find no reason to correct their behavior because the path allows both the Keynes-Ramsey rule and the transversality condition to be satisfied.

No other path than the saddle path can be an equilibrium path. This is because no other technically feasible path is compatible with the households' individual utility maximization under perfect foresight. An initial point above point A can be excluded because the implied path of type II does not satisfy the household's NPG condition (and, consequently, not at all the transversality condition).[17] If the individual household expected an evolution of r_t and w_t corresponding to path II, then the household would immediately choose a *lower* level of consumption, that is, the household would *deviate* in order not to suffer the same fate as Charles Ponzi. In fact, *all* the households would react in this way. Thus, path II would not be realized and the expectation that it would, can not be a rational expectation.

Likewise, an initial point below point A can be ruled out because the implied path of type III does not satisfy the household's transversality condition but implies over-saving. Indeed, at some point in the future, say at time t_1, the economy's effective capital-labor ratio would pass the golden rule value so that for all $t > t_1$, $r_t < g + n$. But with a rate of interest permanently below the growth rate of wage income of the household, the present value of human wealth is *infinite*. This motivates a *higher* consumption level than that along the path. Thus, if the household expects an evolution of r_t and w_t corresponding to path III, then the household will immediately *deviate* and choose a higher initial level of consumption. But so will *all* the households react and the expectation that the economy will follow path III can not be rational.

We have presumed $0 < \tilde{k}_0 < \tilde{k}^*$. If instead $\tilde{k}_0 > \tilde{k}^*$, the economy would move along the saddle path *from above*. Paths like VI and V in Fig. 10.2 can be ruled out because they violate the transversality condition and the NPG condition, respectively (in fact, violating the NPG implies violating the TVC as well). With this we have shown:

PROPOSITION 1 Assume (A1) and (A2). Let there be a given $\tilde{k}_0 > 0$. Then the Ramsey model exhibits a unique equilibrium path, characterized by $(\tilde{k}_t,\ \tilde{c}_t)$

[17] This is shown in Appendix C.

converging, for $t \to \infty$, toward a unique steady state with an effective capital-labor ratio, $\tilde{k}^*$, satisfying $f'(\tilde{k}^*) - \delta = \rho + \theta g$. In the steady state the real interest rate is given by the modified-golden-rule formula, $r^* = \rho + \theta g$, the per capita consumption path is $c_t^* = \tilde{c}^* T_0 e^{gt}$, where $\tilde{c}^* = f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*$, and the real wage path is $w_t^* = \tilde{w}(\tilde{k}^*) T_0 e^{gt}$.

A numerical example based on one year as the time unit: $g = 0.02$, $n = 0.01$, $\theta = 2$, and $\rho = 0.01$. Then, $r^* = 0.05 > 0.03 = g + n$.

So output per capita, $y_t \equiv Y_t/L_t \equiv \tilde{y}_t T_t$, tends to grow at the rate of technological progress, g :

$$\frac{\dot{y}_t}{y_t} \equiv \frac{\dot{\tilde{y}}_t}{\tilde{y}_t} + \frac{\dot{T}_t}{T_t} = \frac{f'(\tilde{k}_t)\dot{\tilde{k}}_t}{f(\tilde{k}_t)} + g \to g \quad \text{for} \quad t \to \infty,$$

in view of $\dot{\tilde{k}}_t \to 0$ combined with $\lim_{t\to\infty} f'(\tilde{k}_t)/f(\tilde{k}_t) = f'(\tilde{k}^*)/f(\tilde{k}^*)$. This is also true for the growth rate of consumption per capita and the real wage, since $c_t \equiv \tilde{c}_t T_t$ and $w_t = \tilde{w}(\tilde{k}_t) T_t$.

The intuition behind the convergence lies in the neoclassical principle of *diminishing marginal productivity of capital.* Starting from a *low* effective capital-labor ratio and therefore a high marginal and average productivity of capital, the resulting high aggregate saving[18] will be more than enough to maintain the effective capital-labor ratio which therefore increases. But when this happens, the marginal and average productivity of capital decreases and the resulting saving, as a proportion of the capital stock, declines until eventually it is only sufficient to replace worn-out machines and equip new "effective" workers with enough machines to just maintain the effective capital-labor ratio. If instead we start from a *high* effective capital-labor, a similar story can be told in reverse. In the long run the effective capital-labor ratio settles down at the steady-state level, $\tilde{k}^*$, where the marginal saving and investment yields a return as great as the representative household's willingness to postpone the marginal unit of consumption. Since the adjustment process is based on capital accumulation, the process is slow. The "speed of adjustment", in the sense of the proportionate rate of decline per year of the distance to the steady state, $\left|\tilde{k} - \tilde{k}^*\right|$, is generally assessed to be in the interval (0.02, 0.10), assuming absence of disturbances to the system during the adjustment.

The equilibrium path generated by the Ramsey model is necessarily dynamically efficient and satisfies the modified golden rule in the long run. Why is there this contrast to Diamonds OLG model where equilibrium paths *may* be

[18] Saving will be high because the negative substitution and wealth effects on current consumption of the high interest rate dominate the income effect.

dynamically inefficient? The reason lies in the fact that only a "single infinity", not a "double infinity", is present in the Ramsey model. The time horizon of the economy is infinite but the number of decision makers is finite. Births (into adult life) do not reflect the emergence of new economic agents with separate interests. In the OLG model, however, births imply entrance of new economic decision makers whose preferences no-one cared about in advance. In that model neither is there any final date, nor any final decision maker. Because of this difference, in several respects the two models give different results. A type of equilibria, namely dynamically inefficient ones, can be realized in the Diamond model but not so in the Ramsey model. A rate of time preference low enough to generate a *tendency* to a long-run interest rate below the income growth rate is inconsistent with existence of general equilibrium in the Ramsey model. It was precisely with the aim of ruling out such a low rate of impatience that we imposed the parameter restriction (A1) above.

The concept of saddle-point stability

The steady state of the model is globally asymptotically stable for arbitrary initial values of the effective capital-labor ratio (the phase diagram only verifies local asymptotic stability, but the extension to global asymptotic stability is verified in Appendix A). If $\tilde{k}$ is hit by a shock at time 0 (say by a discrete jump in the technology level T_0), the economy will converge toward the same unique steady state as before. At first glance this might seem peculiar considering that the steady state is a saddle point. Such a steady state is unstable for arbitrary values of *both* coordinates of the initial point $(\tilde{k}_0, \tilde{c}_0)$. But the crux of the matter is that it is only the initial $\tilde{k}$ that *is* arbitrary. The model assumes that the decision variable c_0, and therefore the value of $\tilde{c}_0 \equiv c_0/T_0$, immediately adjusts to the new situation. That is, the model assumes that $\tilde{c}_0$ always takes the value needed for the household's transversality condition under perfect foresight to be satisfied. This ensures that the economy is initially on the saddle path, cf. the point A in Fig. 10.2. In the language of differential equations *conditional* asymptotic stability is present. The condition that transform the conditional stability to actual stability is the transversality condition.

We shall follow the common terminology in macroeconomics and call a steady state of a two-dimensional dynamic system (locally) *saddle-point stable* if:

1. the steady state is a saddle point;

2. one of the two endogenous variables is predetermined while the other is a jump variable;

3. at least close to the steady state, the saddle path is not parallel to the jump-variable axis;

4. there is a boundary condition on the system such that the diverging paths are ruled out as solutions.

To establish saddle-point stability, all four properties must be verified. If for instance point 1 and 2 hold but, contrary to point 3, the saddle path is parallel to the jump variable axis, then saddle-point stability does not obtain. Indeed, given that the predetermined variable initially deviated from its steady-state value, it would not be possible to find any initial value of the jump variable such that the solution of the system would converge to the steady state for $t \to \infty$.

In the present case, we have already verified point 1 and 2. And as the phase diagram indicates, the saddle path is not vertical. So also point 3 holds. The transversality condition ensures that also point 4 holds. Thus, the Ramsey model is saddle-point stable. In Appendix A it is shown that the positively-sloped saddle path in Fig. 10.2 ranges over *all* $\tilde{k} > 0$ (there is nowhere a vertical asymptote to the saddle path). Hence, the steady state is *globally* saddle-point stable. All in all, these characteristics of the Ramsey model are analogue to those of Barro's dynasty model in discrete time when the bequest motive is operative.

10.4 Comparative analysis

10.4.1 The role of key parameters

The conclusion that in the long run the real interest rate is given by the modified golden rule formula, $r^* = \rho + \theta g$, tells us that only three parameters matter: the rate of time preference, the elasticity of marginal utility, and the rate of technological progress. A higher ρ, i.e., more impatience and thereby less willingness to defer consumption, implies less capital accumulation and thus in the long run smaller effective capital-labor ratio, higher interest rate, and lower consumption than otherwise. The long-run growth rate is unaffected.

A higher θ will have a similar effect, when $g > 0$. As θ is a measure of the desire for consumption smoothing, a higher θ implies that a larger part of the greater wage income in the future, reflecting technology growth, will be consumed immediately. This implies less saving and thereby less capital accumulation and so a lower $\tilde{k}^*$ and higher r^*. Similarly, the long-run interest rate will depend positively on the technology growth rate g because the higher g is, the greater is the expected future wage income. Thereby the consumption possibilities in the future are greater even without any current saving. This discourages current saving and we end up with lower capital accumulation and lower effective capital-labor ratio in the long run, hence higher interest rate. It is also true that the higher is g, the higher is the rate of return needed to induce the saving required for maintaining a steady state and resist the desire for more consumption smoothing.

The long-run interest rate is independent of the particular form of the aggregate production function, f. This function matters for *what* effective capital-labor ratio and *what* consumption level per unit of effective labor are compatible with the long-run interest rate. This kind of results are specific to representative agent models. This is because only in these models will the Keynes-Ramsey rule hold not only for the individual household, but also at the aggregate level.

Unlike the Solow growth model, the Ramsey model provides a *theory* of the evolution and long-run level of the saving-income ratio. The endogenous saving-income ratio of the economy is

$$\begin{aligned} s_t &\equiv \frac{Y_t - C_t}{Y_t} = \frac{\dot{K}_t + \delta K_t}{Y_t} = \frac{\dot{K}_t / K_t + \delta}{Y_t / K_t} = \frac{\dot{\tilde{k}}_t / \tilde{k}_t + g + n + \delta}{f(\tilde{k}_t)/\tilde{k}_t} \\ &\rightarrow \frac{g + n + \delta}{f(\tilde{k}^*)/\tilde{k}^*} \equiv s^* \quad \text{for} \quad t \rightarrow \infty. \end{aligned} \tag{10.38}$$

By determining the path of $\tilde{k}_t$, the Ramsey model determines how s_t moves over time and adjusts to its constant long-run level. Indeed, for any given $\tilde{k} > 0$, the equilibrium value of $\tilde{c}_t$ is uniquely determined by the requirement that the economy must be on the saddle path. Since this defines $\tilde{c}_t$ as a function, $\tilde{c}(\tilde{k}_t)$, of $\tilde{k}_t$, there is a corresponding function for the saving-income ratio in that $s_t = 1 - \tilde{c}(\tilde{k}_t)/f(\tilde{k}_t) \equiv s(\tilde{k}_t)$. So $s(\tilde{k}^*) = s^*$.

We note that the long-run saving-income ratio is a decreasing function of the rate of impatience, ρ, and the desire of consumption smoothing, θ. The ratio is an increasing function of the capital depreciation rate, δ, and the rate of population growth, n.

For an example with an explicit formula for the long-run saving-income ratio, consider:

EXAMPLE 1 Suppose the production function is Cobb-Douglas:

$$\tilde{y} = f(\tilde{k}) = A\tilde{k}^{\alpha}, \qquad A > 0, 0 < \alpha < 1. \tag{10.39}$$

Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1} = \alpha f(\tilde{k})/\tilde{k}$. In steady state we get, by use of the steady-state result (10.34),

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{1}{\alpha} f'(\tilde{k}^*) = \frac{\delta + \rho + \theta g}{\alpha}.$$

Substitution in (10.38) gives

$$s^* = \alpha \frac{\delta + g + n}{\delta + \rho + \theta g} < \alpha, \tag{10.40}$$

where the inequality follows from our parameter restriction (A1). Indeed, (A1) implies $\rho + \theta g > g + n$. The long-run saving-income ratio depends positively on the following parameters: the elasticity of production w.r.t. to capital, α, the capital depreciation rate, δ, and the population growth rate, n. The long-run saving-income ratio depends negatively on the rate of impatience, ρ, and the desire for consumption smoothing, θ. The role of the rate of technological progress is ambiguous.[19]

A numerical example (time unit = 1 year): If $n = 0.005$, $g = 0.015$, $\rho = 0.025$, $\theta = 3$, and $\delta = 0.07$, then $s^* = 0.21$. With the same parameter values except $\delta = 0.05$, we get $s^* = 0.19$.

It can be shown (see Appendix D) that if, by coincidence, $\theta = 1/s^*$, then $s'(\tilde{k}) = 0$, that is, the saving-income ratio s_t is also outside of steady state equal to s^*. In view of (10.40), the condition $\theta = 1/s^*$ is equivalent to the "knife-edge" condition $\theta = (\delta + \rho)/\left[\alpha(\delta + g + n) - g\right] \equiv \bar{\theta}$. More generally, assuming $\alpha(\delta + g + n) > g$ (which seems likely empirically), we have that if $\theta \lesseqgtr 1/s^*$ (i.e., $\theta \lesseqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqgtr 0$, respectively (and if instead $\alpha(\delta + g + n) \leq g$, then $s'(\tilde{k}) < 0$, unconditionally).[20] Data presented in Barro and Sala-i-Martin (2004, p. 15) indicate no trend for the US saving-income ratio, but a positive trend for several other developed countries since 1870. One interpretation is that whereas the US has for a long time been close to its steady state, the other countries are still in the adjustment process toward the steady state. As an example, consider the parameter values $\delta = 0.05$, $\rho = 0.02$, $g = 0.02$ and $n = 0.01$. In this case we get $\bar{\theta} = 10$ if $\alpha = 0.33$; given $\theta < 10$, these other countries should then have $s'(\tilde{k}) < 0$ which, according to the model, is compatible with a rising saving-income ratio over time only if these countries are approaching their steady state from *above* (i.e., they should have $\tilde{k}_0 > \tilde{k}^*$). It may be argued that α should also reflect the role of education and R&D in production and thus be higher; with $\alpha = 0.75$ we get $\bar{\theta} = 1.75$. Then, if $\theta > 1.75$, these countries would have $s'(\tilde{k}) > 0$ and thus approach their steady state from *below* (i.e., $\tilde{k}_0 < \tilde{k}^*$). □

10.4.2 Special case: Solow's growth model*

The above results give a hint that Solow's growth model, with a given constant saving-income ratio $s \in (0, 1)$ and given δ, g, and n (with $\delta + g + n > 0$), can, under

[19] Partial differentiation w.r.t. g yields $\partial s^*/\partial g = \alpha\left[\rho - \theta n - (\theta - 1)\delta\right]/(\delta + \rho + \theta g)^2$, the sign of which cannot be determined a priori.

[20] See Appendix D.

certain circumstances, be interpreted as a special case of the Ramsey model. The Solow model in continuous time is given by

$$\dot{\tilde{k}}_t = sf(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t.$$

The constant saving-income ratio implies proportionality between consumption and income. In growth-corrected terms per capita consumption is

$$\tilde{c}_t = (1 - s)f(\tilde{k}_t).$$

For the Ramsey model to yield this, the production function must be like in (10.39) (i.e., Cobb-Douglas) with $\alpha > s$. And the elasticity of marginal utility, θ, must satisfy $\theta = 1/s$. Finally, the rate of time preference, ρ, must be such that (10.40) holds with s^* replaced by s, which implies $\rho = \alpha(\delta + g + n)/s - \delta - \theta g$. It remains to show that this ρ satisfies the inequality, $\rho - n > (1 - \theta)g$, which is necessary for existence of an equilibrium in the Ramsey model. Since $\alpha/s > 1$, the chosen ρ satisfies $\rho > \delta + g + n - \delta - \theta g = n + (1 - \theta)g$, which was to be proved. Thus, in this case the Ramsey model generates an equilibrium path which implies an evolution identical to that generated by the Solow model with $s = 1/\theta$.[21]

With this foundation of the Solow model, it will always hold that $s = s^* < s_{GR}$, where s_{GR} is the golden rule saving-income ratio. Indeed, from (10.38) and (10.32), respectively,

$$s_{GR} = \frac{(\delta + g + n)\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \frac{f'(\tilde{k}_{GR})\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \alpha > s^*,$$

from the Cobb-Douglas specification and (10.40), respectively.

A point of the Ramsey model vis-a-vis the Solow model is to replace a mechanical saving rule by maximization of discounted utility and thereby, on the one hand, open up for (i) a wider range of possible evolutions; (ii) welfare analysis; and (iii) analysis of incentive effects of economic policy on households' saving. On the other hand, in some respects the Ramsey model narrows down the range of possibilities, for example by unconditionally ruling out over-accumulation (dynamic inefficiency).

10.5 A social planner's problem

Another implication of the Ramsey framework is that the decentralized market equilibrium (within the idealized presumptions of the model) brings about the same allocation of resources as would a social planner facing the same technology and initial resources as described above and having the same criterion function as the representative household.

[21] A proof is given in Appendix D.

10.5.1 The equivalence theorem

As in Chapter 8, by a *social planner* we mean a hypothetical central authority who is "all-knowing and all-powerful" and is constrained only by the limitations arising from technology and initial resources. Within these confines the social planner can fully decide on the resource allocation. Since we consider a closed economy, the social planner has no access to an international loan market.

Let the economy be closed and let the social welfare function be time separable with constant elasticity, $\hat{\theta}$, of marginal utility and a pure rate of time preference $\hat{\rho}$.[22] Then the social planner's optimization problem is

$$\begin{aligned}
\max_{(c_t)_{t=0}^{\infty}} W_0 &= \int_0^{\infty} \frac{c_t^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-(\hat{\rho}-n)t} dt \qquad \text{s.t.} && (10.41)\\
c_t &\geq 0, && (10.42)\\
\dot{\tilde{k}}_t &= f(\tilde{k}_t) - \frac{c_t}{T_t} - (\delta + g + n)\tilde{k}_t, && (10.43)\\
\tilde{k}_t &\geq 0 \qquad \text{for all } t \geq 0. && (10.44)
\end{aligned}$$

We assume $\hat{\theta} > 0$ and $\hat{\rho} - n > (1 - \hat{\theta})g$ in line with the assumption (A1) for the market economy above. In case $\hat{\theta} = 1$, the expression $c_t^{1-\hat{\theta}}/(1-\hat{\theta})$ should be interpreted as $\ln c_t$. No market prices or other elements belonging to the specific market institutions of the economy enter the social planner's problem. The dynamic constraint (10.43) reflects the national product account. Because the economy is closed, the social planner does not have the opportunity of borrowing or lending from abroad. Hence there is no solvency requirement. Instead we just impose the definitional constraint (10.44) of non-negativity of the state variable $\tilde{k}$.

The social planner's problem is to select, within the technically feasible paths, the one that maximizes the value of the social welfare function W_0. The problem is a continuous time analogue of the social planner's problem in discrete time in Chapter 8. Note, however, a minor conceptual difference, namely that in continuous time there is in the short run no *upper* bound on the *flow* variable c_t, that is, no bound like $c_t \leq T_t \left[f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t \right]$. A consumption intensity c_t which is higher than the right-hand side of this inequality will just be reflected in a negative value of the flow variable $\dot{\tilde{k}}_t$.[23]

[22] Possible reasons for allowing these two preference parameters to deviate from the corresponding parameters in the private sector were discussed in Chapter 8.1.1.

[23] As usual we presume that capital can be "eaten". That is, we consider the capital good to be instantaneously convertible to a consumption good. Otherwise there *would* be at any time

To solve the problem we apply the Maximum Principle. The current-value Hamiltonian is

$$H(\tilde{k}, c, \lambda, t) = \frac{c^{1-\hat{\theta}}}{1-\hat{\theta}} + \lambda \left[f(\tilde{k}) - \frac{c}{T} - (\delta + g + n)\tilde{k} \right],$$

where λ is the adjoint variable associated with the dynamic constraint (10.43). An interior optimal path $(\tilde{k}_t, c_t)_{t=0}^{\infty}$ will satisfy that there exists a continuous function $\lambda = \lambda(t)$ such that, for all $t \geq 0$,

$$\frac{\partial H}{\partial c} = c^{-\hat{\theta}} - \frac{\lambda}{T} = 0, \text{ i.e., } c^{-\hat{\theta}} = \frac{\lambda}{T}, \quad \text{and} \tag{10.45}$$

$$\frac{\partial H}{\partial \tilde{k}} = \lambda(f'(\tilde{k}) - \delta - g - n) = (\hat{\rho} - n)\lambda - \dot{\lambda} \tag{10.46}$$

hold along the path. Finally, in the present problem the "standard" transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t \lambda_t e^{-(\hat{\rho}-n)t} = 0, \tag{10.47}$$

is necessary for optimality, when $\hat{\rho} - n > (1 - \hat{\theta})g$, as assumed above.[24]

The condition (10.45) can be seen as a $MC = MB$ condition and illustrates that λ_t is the social planner's shadow price, measured in terms of current utility, of $\tilde{k}_t$ along the optimal path.[25] The differential equation (10.46) tells us how this shadow price evolves over time. The transversality condition, (10.47), together with (10.45), entails the condition

$$\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{gt} e^{-(\hat{\rho}-n)t} = 0,$$

where the unimportant factor T_0 has been eliminated. Imagine the opposite were true, namely that $\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{[g-(\hat{\rho}-n)]t} > 0$. Then, intuitively U_0 could be increased by reducing the long-run value of $\tilde{k}_t$, i.e., consume more and save less.

By taking logs in (10.45) and differentiating w.r.t. t, we get $-\hat{\theta}\dot{c}/c = \dot{\lambda}/\lambda - g$. Inserting (10.46) and rearranging gives the condition

$$\frac{\dot{c}}{c} = \frac{1}{\hat{\theta}}(g - \frac{\dot{\lambda}}{\lambda}) = \frac{1}{\hat{\theta}}(f'(\tilde{k}) - \delta - \hat{\rho}). \tag{10.48}$$

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an upper bound on c, namely $c \leq Tf(\tilde{k})$, saying that the per capita consumption flow cannot exceed the per capita output flow. The role of such constraints is discussed in Feichtinger and Hartl (1986).

[24] See Appendix E.

[25] Decreasing c_t by one unit, increases $\tilde{k}_t$ by $1/T_t$ units, each of which are worth λ_t utility units to the social planner.

This is the social planner's Keynes-Ramsey rule. If the rate of time preference, $\hat{\rho}$, is lower than the net marginal productivity of capital, $f'(\tilde{k})-\delta$, the social planner will let per capita consumption be relatively low in the beginning in order to attain greater per capita consumption later. The lower the impatience relative to the return to capital, the more favorable it becomes to defer consumption.

Because $\tilde{c} \equiv c/T$, we get from (10.48) qualitatively the same differential equation for $\tilde{c}$ as we obtained in the decentralized market economy. And the dynamic resource constraint (10.43) is of course identical to that of the decentralized market economy. Thus, the dynamics are in principle unaltered and the phase diagram in Fig. 10.2 is still valid. The solution of the social planner implies that the economy will move along the saddle path toward the steady state. This trajectory, path I in the diagram, satisfies both the first-order conditions and the transversality condition. However, paths such as III in the figure do not satisfy the transversality condition of the social planner but imply permanent over-saving. And paths such as II in the figure will experience a sudden end when all the capital has been used up. Intuitively, they cannot be optimal. A rigorous argument is given in Appendix E, based on the fact that the Hamiltonian is *strictly concave* in $(\tilde{k}, \tilde{c})$. Thence, not only is the saddle path an optimal solution, it is the *unique* optimal solution.

Comparing with the market solution of the previous section, we have established:

PROPOSITION 2 *(equivalence theorem)* Consider an economy with neoclassical CRS technology and a representative infinitely-lived household with preferences as in (10.3) with $u(c) = c^{1-\theta}/(1-\theta)$. Assume (A1) and (A2). Let there be a given $\tilde{k}_0 > 0$. Then a perfectly competitive market economy brings about the same resource allocation as that brought about by a social planner with the same criterion function as the representative household, i.e., with $\hat{\theta} = \theta$ and $\hat{\rho} = \rho$.

This is a continuous time analogue to the discrete time equivalence theorem of Chapter 8.

The effective capital-labor ratio $\tilde{k}$ in the social planner's solution will not converge toward the golden rule level, $\tilde{k}_{GR}$, but toward a level whose distance to the golden rule level depends on how much $\hat{\rho} + \hat{\theta}g$ exceeds the natural growth rate, $g + n$. Even if society would be able to consume more in the long term if it aimed for the golden rule level, this would not compensate for the reduction in current consumption which would be necessary to achieve it. This consumption is relatively more valuable, the greater is the social planner's effective rate of time preference, $\hat{\rho} - n$. In line with the market economy, the social planner's solution ends up in a *modified golden rule.* In the long term, net marginal productivity of capital is determined by preference parameters and productivity growth and equals $\hat{\rho}+\hat{\theta}g > g+n$. Hereafter, given the net marginal productivity of capital, the effective capital-labor ratio and the level of the consumption path is determined by the production function.

Varieties of generational discounting* In the above analysis the social planner maximizes the sum of discounted per capita utilities *weighted* by generation size. This implies *utilitaristic discounting.* The *effective* utility discount rate, $\rho - n$, varies negatively (one to one) with the population growth rate. Since this corresponds to how the per capita rate of return on saving, $r - n$, is "diluted" by population growth, the net marginal productivity of capital in steady state becomes independent of n, namely equal to $\hat{\rho} + \hat{\theta}g$.

Some textbooks, Blanchard and Fischer (1989) for instance, let the social planner maximize the sum of discounted per capita utilities *without* weighting by generation size. Then the effective utility discount rate is independent of the population growth rate, n. With $\hat{\rho}$ still denoting the pure rate of time preference, the criterion function becomes

$$W_0 = \int_0^\infty \frac{c_t{}^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-\hat{\rho}t} dt.$$

The social planner's solution then converges toward a steady state with net marginal productivity of capital equal to

$$f'(\tilde{k}^*) - \delta = \hat{\rho} + n + \hat{\theta}g. \tag{10.49}$$

Here, an increase in n will imply higher long-run net marginal productivity of capital and lower effective capital-labor ratio, everything else equal.

The representative household in the market economy described by a Ramsey model may of course also have a criterion function in line with this, that is, $U_0 = \int_0^\infty u(c_t)e^{-\rho t}dt.$ Then, the interest rate in the economy will in the long run be $r^* = \rho + n + \theta g$ and so an increase in n will increase r^* and decrease $\tilde{k}^*$.

The more common approach is the utilitaristic accounting, which may be based on the argument: "if more people benefit, so much the better".

10.5.2 Ramsey's original zero discount rate and the overtaking criterion*

It was mostly the perspective of a social planner, rather than the market mechanism, which was at the center of Ramsey's original analysis (Ramsey, 1928). The case considered by Ramsey has $g = n = 0$. Ramsey maintained that the social planner should "not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination" (Ramsey 1928). So Ramsey has $\rho - n = \rho = 0$. Given the instantaneous utility function, u, where $u' > 0, u'' < 0$, and given $\rho = 0$, Ramsey's original problem was: choose $(c_t)_{t=0}^{\infty}$ so as to optimize (in some sense, see below)

$$\begin{aligned} W_0 &= \int_0^{\infty} u(c_t)dt \qquad \text{s.t.} \\ c_t &\geq 0, \\ \dot{k}_t &= f(k_t) - c_t - \delta k_t, \\ k_t &\geq 0 \qquad \text{for all } t \geq 0. \end{aligned}$$

A condition corresponding to our assumption (A1) above does not apply. So the improper integral W_0 will generally not be bounded[26] and Ramsey can not use maximization of W_0 as an optimality criterion. Instead he considers a criterion akin to the overtaking criterion we considered in a discrete time context in Chapter 8. We only have to reformulate this criterion for a continuous time setting.

Let $(c_t)_{t=0}^{\infty}$ be the consumption path associated with an arbitrary technically feasible path and let $(\hat{c}_t)$ be the consumption path associated with our candidate as an optimal path, that is, the path we wish to test for optimality. Define

$$D_T \equiv \int_0^T u(\hat{c}_t)dt - \int_0^T u(c_t)dt. \tag{10.50}$$

Then the feasible path $(\hat{c}_t)_{t=0}^{\infty}$ is *overtaking optimal*, if for any feasible path, $(c_t)_{t=0}^{\infty}$, there exists a number $T' \geq 0$ such that $D_T \geq 0$ for all $T \geq T'$. That is, if for every alternative feasible path, the candidate path has from some date on, cumulative utility up to *all* later dates at least as great as that of the alternative feasible path, then the candidate path is overtaking optimal.

We say that the candidate path is *weakly preferred* in case we just know that $D_T \geq 0$ for all $T \geq T'$. If $D_T \geq 0$ can be replaced by $D_T > 0$, we say it is *strictly preferred*.[27]

Optimal control theory is also applicable for this criterion. The current-value HamiItonian is

$$H(k, c, \lambda, t) = u(c) + \lambda \left[f(k) - c - \delta k \right].$$

The Maximum Principle states that an interior overtaking-optimal path will satisfy that there exists an adjoint variable λ such that for all $t \geq 0$ it holds along

[26] Suppose for instance that $c_t \to \bar{c}$ for $t \to \infty$. Then $\int_0^{\infty} u(c_t)dt = \pm\infty$ for $u(\bar{c}) \gtrless 0$, respectively.

[27] A slightly more generally applicable optimality criterion is the *catching-up* criterion. The meaning of this criterion in continuous time is analogue to its meaning in discrete time, cf. Chapter 8.3. The overtaking as well as the catching-up criterion entail generally only a *partial* ordering of alternative technically feasible paths.

this path that

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0, \text{ and} \quad (10.51)$$

$$\frac{\partial H}{\partial k} = \lambda(f'(k) - \delta) = -\dot{\lambda}. \quad (10.52)$$

Since $\rho = 0$, the Keynes-Ramsey rule reduces to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(f'(k_t) - \delta), \qquad \text{where } \theta(c) \equiv -\frac{c}{u'(c)}u''(c).$$

One might conjecture that also the transversality condition,

$$\lim_{t\to\infty} k_t\lambda_t = 0, \quad (10.53)$$

is necessary for optimality but, as we will see below, this turns out to be wrong in this case with no discounting.

Our assumption (A2') here reduces to $\lim_{k\to 0} f'(k) > \delta > \lim_{k\to\infty} f'(k)$ (which requires $\delta > 0$). Apart from this, the phase diagram is fully analogue to that in Fig. 10.2, except that the steady state, E, is now at the top of the $\dot{k} = 0$ curve. This is because in steady state, $f'(k^*) - \delta = 0$. This equation also defines k_{GR} in this case. It can be shown that the saddle path is again the unique solution to the optimization problem (the method is essentially the same as in the discrete time case of Chapter 8). The intuitive background is that failing to approach the golden rule would imply a forgone "opportunity of infinite gain".

A noteworthy feature is that in this case the Ramsey model constitutes a counterexample to the widespread presumption that an optimal plan with infinite horizon *must* satisfy a transversality condition like (10.53). Indeed, by (10.51), $\lambda_t = u'(c_t) \to u'(c^*)$ for $t \to \infty$ along the overtaking-optimal path (the saddle path). Thus, instead of (10.53), we get

$$\lim_{t\to\infty} k_t\lambda_t = k^*u'(c^*) > 0.$$

With CRRA utility it is straightforward to generalize these results to the case $g \geq 0, n \geq 0$ and $\hat{\rho} - n = (1 - \hat{\theta})g$. The social planner's overtaking-optimal solution is still the saddle path approaching the golden rule steady state. And this solution violates the seemingly "natural" transversality condition, (10.47), which *is* necessary for optimality when $\hat{\rho} - n > (1 - \hat{\theta})g$, as in Section 10.5.1.

Note also that with zero effective utility discounting, there can not be equilibrium in the *market* economy version of this story. The real interest rate would in the long run be zero and thus the human wealth of the infinitely-lived household would be infinite. But then the demand for consumption goods would be unbounded and equilibrium thus be impossible.

10.6 Concluding remarks

The Ramsey model has played an important role as a way of structuring economists' thoughts about an array of macrodynamic phenomena. The popularity of the model probably derives from the fact that is allows taking microeconomic principles into account without worrying about the usual aggregation problems when going from micro to macro.

As illustrated in Fig. 10.3, the Ramsey model can be seen as situated at one end of a line segment where the Diamond OLG model is situated at the opposite end. Both models build on idealized assumptions. The Diamond model ignores any bequest motive and emphasizes life-cycle behavior and heterogeneity in the population. The Ramsey model implicitly assumes an altruistic bequest motive which is always operative and which turns households into homogeneous, infinitely-lived agents. In this way the Ramsey model ends up as an easy-to-apply framework, suggesting *inter alia* a clear-cut theory of the level of the real interest rate in the long run − the *modified golden rule*. Although this theory finds little empirical support (Hamilton et al., 2016), it facilitates general equilibrium analysis of an array of dynamic problems. The next chapter discusses some examples: effects of unanticipated and anticipated changes in taxation and endogenous growth theory.

The assumption of a representative household is a main limitation. The lack of heterogeneity in the model's population of households implies a danger that important interdependencies between different classes of agents are unduly neglected. For some problems these interdependencies may be of only secondary importance, but they are crucial for others (for instance, issues concerning public debt or interaction between private debtors and creditors). On the other hand, as Caselli and Ventura (2000) have shown, it is possible to extend the Ramsey model so as to allow heterogeneity in the population with respect to initial financial wealth and labor productivity. But regarding preferences only very limited heterogeneity can be embraced by the Ramsey model.

Another disputed feature of the model is that it endows the households with an extreme amount of information about the future. Solow (1990, p. 221) warns against overly reliance on saddle-point stability in the analysis of a market economy:

> "The problem is not just that perfect foresight into the indefinite future is so implausible away from steady states. The deeper problem is that in practice − if there is any practice − miscalculations about the equilibrium path may not reveal themselves for a long time. The mistaken path gives no signal that it will be "ultimately" infeasible. It is natural to comfort oneself: whenever the error is perceived there will be a jump to a better

> approximation to the converging arm. But a large jump may be required. In a decentralized economy it will not be clear who knows what, or where the true converging arm is, or, for that matter, exactly where we are now, given that some agents (speculators) will already have perceived the need for a mid-course correction while others have not. This thought makes it hard even to imagine what a long-run path would look like. It strikes me as more or less devastating for the interpretation of quarterly data as the solution of an infinite time optimization problem."

As we saw in Section 10.5.2, Ramsey's original analysis (Ramsey 1928) dealt with a social planner's infinite horizon optimal control problem. In that optimization problem there are well-defined shadow prices, as implied by an explicit social welfare function. In a decentralized market economy, however, there are a multitude of both agents and prices and no god-like auctioneer to ensure that the long-term price expectations coincide with the long-term shadow prices in the social planner's optimal control problem.

Fig. 10.3 about here (not yet available)

While the Ramsey and the Diamond model are polar cases along the line segment in Fig. 10.3, less abstract macro models are scattered between these poles, some being closer to one pole than to the other. Sometimes a given model open up for alternative *regimes*, one close to Ramsey's pole, another close to Diamond's. An example is Robert Barro's model with parental altruism discussed in Chapter 7. When the bequest motive in the Barro model is operative, the model coincides with a Ramsey model (in discrete time) as was shown in Chapter 8. But when the bequest motive is not operative, the Barro model coincides with a Diamond OLG model. This conditionality "places" the Barro model in the interior of the line segment, but in practice closer to Ramsey's pole than to Diamond's. model

Blanchard's OLG model in continuous time (to be analyzed and used in chapters 12, 13, and 15) also belongs to the interior of the line segment, but closer to Diamond's pole than to Ramsey's.

10.7 Literature notes

1. Frank Ramsey (1903-1930) died at the age of 26 but he managed to publish several path-breaking articles in economics. Ramsey discussed economic issues

with, among others, John Maynard Keynes. In an obituary published in the Economic Journal (March 1932) after Ramsey's death, Keynes described Ramsey's article about the optimal savings as "one of the most remarkable contributions to mathematical economics ever made, both in respect of the intrinsic importance and difficulty of its subject, the power and elegance of the technical methods employed, and the clear purity of illumination with which the writer's mind is felt by the reader to play about its subject".

2. The version of the Ramsey model we have considered is in accordance with the general tenet of neoclassical preference theory: saving is motivated only by higher consumption in the future. Extended versions assume that accumulation of wealth is to some extend an end in itself or perhaps motivated by a desire for social prestige and economic and political power rather than consumption. In Kurz (1968b) an extended Ramsey model is studied where wealth is an independent argument in the instantaneous utility function.

Also Tournemaine and Tsoukis (2008) and Long and Shimomura (2004).

3. The equivalence in the Ramsey model between the decentralized market equilibrium and the social planner's solution can be seen as an extension of the first welfare theorem as it is known from elementary textbooks, to the case where the market structure stretches infinitely far out in time, and the finite number of economic agents (family dynasties) face an infinite time horizon: in the absence of externalities etc., the allocation of resources under perfect competition will lead to a Pareto optimal allocation. The Ramsey model is indeed a special case in that all households are identical. But the result can be shown in a far more general setup, cf. Debreu (1954). The result, however, does not hold in overlapping generations models where an unbounded sequence of new generations enter and the "interests" of the new households have not been accounted for in advance.

4. The simple counter-example to the "standard" necessary transversality condition for an infinite horizon optimal control problem given in Section 10.5.2 was for a problem where the utility integral was not bounded from above. The optimality criterion was therefore not maximization but overtaking. But Sydsæter et al. (2008) contains a counter-example for a problem where maximization *is* the optimality criterion.

10.8 Appendix

A. Algebraic analysis of the dynamics around the steady state

To supplement the graphical approach of Section 10.3 with an exact analysis of the adjustment dynamics of the model, we compute the Jacobian matrix for the system of differential equations (10.28) - (10.29):

$$J(\tilde{k},\tilde{c}) = \begin{bmatrix} \partial\dot{\tilde{k}}/\partial\tilde{k} & \partial\dot{\tilde{k}}/\partial\tilde{c} \\ \partial\dot{\tilde{c}}/\partial\tilde{k} & \partial\dot{\tilde{c}}/\partial\tilde{c} \end{bmatrix} = \begin{bmatrix} f'(\tilde{k}) - (\delta + g + n) & -1 \\ \frac{1}{\theta}f''(\tilde{k})\tilde{c} & \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho + \theta g) \end{bmatrix}.$$

Evaluated in the steady state this reduces to

$$J(\tilde{k}^*,\tilde{c}^*) = \begin{bmatrix} \rho - n - (1-\theta)g & -1 \\ \frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* & 0 \end{bmatrix}$$

This matrix has the determinant

$$\frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* < 0.$$

Since the product of the eigenvalues of the matrix equals the determinant, the eigenvalues are real and opposite in sign.

In standard math terminology a steady-state point in a two dimensional continuous-time dynamic system is called a *saddle point* if the associated eigenvalues are opposite in sign.[28] For the present case we conclude that the steady state is a saddle point. This mathematical definition of a saddle point is equivalent to that given in the text of Section 10.3. Indeed, with two eigenvalues of opposite sign, there exists, in a small neighborhood of the steady state, a stable arm consisting of two saddle paths which point in opposite directions. From the phase diagram in Fig. 10.2 we know that the stable arm has a positive slope. At least for $\tilde{k}_0$ sufficiently close to $\tilde{k}^*$ it is thus possible to start out on a saddle path. Consequently, there is a (unique) value of $\tilde{c}_0$ such that $(\tilde{k}_t, \tilde{c}_t) \to (\tilde{k}^*, \tilde{c}^*)$ for $t \to \infty$. Finally, the dynamic system has exactly one jump variable, $\tilde{c}$, and one predetermined variable, $\tilde{k}$. It follows that the steady state is (locally) *saddle-point stable.*

We claim that for the present model this can be strengthened to *global* saddle-point stability. Indeed, for *any* $\tilde{k}_0 > 0$, it is possible to start out on the saddle path. For $0 < \tilde{k}_0 \leq \tilde{k}^*$, this is obvious in that the extension of the saddle path toward the left reaches the y-axis at a non-negative value of $\tilde{c}^*$. That is to say that the extension of the saddle path cannot, according to the uniqueness theorem for differential equations, intersect the $\tilde{k}$-axis for $\tilde{k} > 0$ in that the positive part of the $\tilde{k}$-axis is a solution of (10.28) - (10.29).[29]

[28] Note the difference compared to a discrete time system, cf. Appendix D of Chapter 8. In the discrete time system we have next period's $\tilde{k}$ and $\tilde{c}$ on the left-hand side of the dynamic equations, not the increase in $\tilde{k}$ and $\tilde{c}$, respectively. Therefore, the criterion for a saddle point looks different in discrete time.

[29] Because the extension of the saddle path towards the left in Fig. 10.1 can not intersect the $\tilde{c}$-axis at a value of $\tilde{c} > f(0)$, it follows that if $f(0) = 0$, the extension of the saddle path ends up in the origin.

For $\tilde{k}_0 > \tilde{k}^*$, our claim can be verified in the following way: suppose, contrary to our claim, that there exists a $\tilde{k}_1 > \tilde{k}^*$ such that the saddle path does not intersect that region of the positive quadrant where $\tilde{k} \geq \tilde{k}_1$. Let $\tilde{k}_1$ be chosen as the smallest possible value with this property. The slope, $d\tilde{c}/d\tilde{k}$, of the saddle path will then have no upper bound when $\tilde{k}$ approaches $\tilde{k}_1$ from the left. Instead $\tilde{c}$ will approach ∞ along the saddle path. But then $\ln \tilde{c}$ will also approach ∞ along the saddle path for $\tilde{k} \to \tilde{k}_1$ ($\tilde{k} < \tilde{k}_1$). It follows that $d \ln \tilde{c}/d\tilde{k} = (d\tilde{c}/d\tilde{k})/\tilde{c}$, computed along the saddle path, will have no upper bound. Nevertheless, we have

$$\frac{d \ln \tilde{c}}{d\tilde{k}} = \frac{d \ln \tilde{c}/dt}{d\tilde{k}/dt} = \frac{\dot{\tilde{c}}/\tilde{c}}{\dot{\tilde{k}}} = \frac{\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho - \theta g)}{f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}}.$$

When $\tilde{k} \to \tilde{k}_1$ and $\tilde{c} \to \infty$, the numerator in this expression is bounded, while the denominator will approach $-\infty$. Consequently, $d \ln \tilde{c}/d\tilde{k}$ will approach zero from above, as $\tilde{k} \to \tilde{k}_1$. But this contradicts that $d \ln \tilde{c}/d\tilde{k}$ has no upper bound, when $\tilde{k} \to \tilde{k}_1$. Thus, the assumption that such a $\tilde{k}_1$ exists is false and our original hypothesis holds true.

B. Boundedness of the utility integral

We claimed in Section 10.3 that if the parameter restriction

$$\rho - n > (1 - \theta)g \tag{A1}$$

holds, then the utility integral, $U_0 = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt$, is bounded, from above as well as from below, along the steady-state path, $c_t = \tilde{c}^* T_t$. The proof is as follows. Recall that $\theta > 0$ and $g \geq 0$. For $\theta \neq 1$,

$$\begin{aligned} (1-\theta)U_0 &= \int_0^\infty c_t^{1-\theta} e^{-(\rho-n)t} dt = \int_0^\infty (c_0 e^{gt})^{1-\theta} e^{-(\rho-n)t} dt \\ &= c_0 \int_0^\infty e^{[(1-\theta)g-(\rho-n)]t} dt = \frac{c_0}{\rho - n - (1-\theta)g}, \end{aligned} \tag{10.54}$$

which by (A1) is finite and positive since $c_0 > 0$. If $\theta = 1$, so that $u(c) = \ln c$, we get

$$U_0 = \int_0^\infty (\ln c_0 + gt) e^{-(\rho-n)t} dt, \tag{10.55}$$

which is also finite, in view of (A1) implying $\rho - n > 0$ in *this* case (the exponential term, $e^{-(\rho-n)t}$, declines faster than the linear term gt increases). It follows that also any path converging to the steady state will entail bounded utility, when (A1) holds.

On the other hand, suppose that (A1) does *not* hold, i.e., $\rho - n \leq (1-\theta)g$. Then by the third equality in (10.54) and $c_0 > 0$ follows that $(1-\theta)U_0 = \infty$ if $\theta \neq 0$. If instead $\theta = 1$, (10.55) implies $U_0 = \infty$.

C. The diverging paths

In Section 10.3 we stated that paths of types *II* and *III* in the phase diagram in Fig. 10.2 can not be equilibria with perfect foresight. Given the expectation corresponding to any of these paths, every single household will choose to *deviate* from the expected path (i.e., deviate from the expected "average behavior" in the economy). We will now show this formally.

We first consider a path of type *III.* A path of this type will not be able to *reach* the horizontal axis in Fig. 10.2. It will only *converge* toward the point $(\bar{\tilde{k}}, 0)$ for $t \to \infty$. This claim follows from the uniqueness theorem for differential equations with continuously differentiable right-hand sides. The uniqueness implies that two solution curves cannot intersect. And we see from (10.29) that the positive part of the x-axis is from a mathematical point of view a solution curve (and the point $(\bar{\tilde{k}}, 0)$ is a trivial steady state). This rules out another solution curve hitting the x-axis.

The convergence of $\tilde{k}$ toward $\bar{\tilde{k}}$ implies $\lim_{t\to\infty} r_t = f'(\bar{\tilde{k}}) - \delta < g + n$, where the inequality follows from $\bar{\tilde{k}} > \tilde{k}_{GR}$. So,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = \lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (r_s - g - n)ds} = \lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = \bar{\tilde{k}} e^{\infty} > 0. \tag{10.56}$$

Hence the transversality condition of the households is violated. Consequently, the household will choose higher consumption than along this path and can do so without violating the NPG condition.

Consider now instead a path of type *II.* We shall first show that if the economy follows such a path, then depletion of all capital occurs in finite time. Indeed, in the text it was shown that any path of type *II* will pass the $\dot{\tilde{k}} = 0$ locus in Fig. 10.2. Let t_0 be the point in time where this occurs. If path *II* lies above the $\dot{\tilde{k}} = 0$ locus for all $t \geq 0$, then we set $t_0 = 0$. For $t > t_0$, we have

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t < 0.$$

By differentiation w.r.t. t we get

$$\ddot{\tilde{k}}_t = f'(\tilde{k}_t)\dot{\tilde{k}}_t - \dot{c}_t - (\delta + g + n)\dot{\tilde{k}}_t = [f'(\tilde{k}_t) - \delta - g - n]\dot{\tilde{k}}_t - \dot{c}_t < 0,$$

where the inequality comes from $\dot{\tilde{k}}_t < 0$ combined with the fact that $\tilde{k}_t < \tilde{k}_{GR}$ implies $f'(\tilde{k}_t) - \delta > f'(\tilde{k}_{GR}) - \delta = g + n$. Therefore, there exists a $t_1 > t_0 \geq 0$ such that

$$\tilde{k}_{t_1} = \tilde{k}_{t_0} + \int_{t_0}^{t_1} \dot{\tilde{k}}_t dt = 0,$$

as was to be shown. At time t_1, $\tilde{k}$ cannot fall any further and $\tilde{c}_t$ immediately drops to $f(0)$ and stay there hereafter.

Yet, this result does not in itself explain why the individual household will deviate from such a path. The individual household has a negligible impact on the movement of $\tilde{k}_t$ in society and correctly perceives r_t and w_t as essentially independent of its own consumption behavior. Indeed, the economy-wide $\tilde{k}$ is not the household's concern. What the household cares about is its own financial wealth and budget constraint. In the perspective of the household nothing prevents it from planning a negative financial wealth, a, and possibly a continuously declining financial wealth, if only the NPG condition,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n) ds} \geq 0,$$

is satisfied.

But we can show that paths of type *II* will *violate* the NPG condition. The reasoning is as follows. The household plans to follow the Keynes-Ramsey rule. Given an expected evolution of r_t and w_t corresponding to path *II*, this will imply a planned gradual transition from positive financial wealth to debt. The transition to positive net debt, $\tilde{d}_t \equiv -\tilde{a}_t \equiv -a_t/T_t > 0$, takes place at time t_1 defined above.

The continued growth in the debt will meanwhile be so fast that the NPG condition is violated. To see this, note that the NPG condition implies the requirement

$$\lim_{t\to\infty} \tilde{d}_t e^{-\int_0^t (r_s - g - n) ds} \leq 0, \tag{NPG}$$

that is, the productivity-corrected debt, $\tilde{d}_t$, is allowed to grow in the long run only at a rate *less* than the growth-corrected real interest rate. For $t > t_1$ we get from the accounting equation $\dot{a}_t = (r_t - n)a_t + w_t - c_t$ that

$$\dot{\tilde{d}}_t = (r_t - g - n)\tilde{d}_t + \tilde{c}_t - \tilde{w}_t > 0,$$

where $\tilde{d}_t > 0$, $r_t > \rho + \theta g > g + n$, and where $\tilde{c}_t$ grows exponentially according to the Keynes-Ramsey rule, while $\tilde{w}_t$ is non-increasing in that $\tilde{k}_t$ does not grow. This implies

$$\lim_{t\to\infty} \frac{\dot{\tilde{d}}_t}{\tilde{d}_t} \geq \lim_{t\to\infty} (r_t - g - n),$$

which is in conflict with (NPG).

Consequently, the household will choose a lower consumption path and thus *deviate* from the reference path considered. Every household will do this and the evolution of r_t and w_t corresponding to path *II* is thus *not* an equilibrium with perfect foresight.

The conclusion is that all individual households understand that the only evolution which can be expected rationally is the one corresponding to the saddle path.

D. Constant saving-income ratio as a special case

As we noted in Section 10.4, Solow's growth model can be seen as a special case of the Ramsey model. Indeed, a constant saving-income ratio may, under certain conditions, emerge as an endogenous result in the Ramsey model.

Let the rate of saving, $(Y_t - C_t)/Y_t$, be s_t. We have generally

$$\tilde{c}_t = (1 - s_t)f(\tilde{k}_t), \qquad \text{and so} \tag{10.57}$$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t = s_t f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t. \tag{10.58}$$

In the Solow model the rate of saving is a constant, s, and we then get, by differentiating with respect to t in (10.57) and using (10.58),

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = f'(\tilde{k}_t)[s - \frac{(\delta + g + n)\tilde{k}_t}{f(\tilde{k}_t)}]. \tag{10.59}$$

By maximization of discounted utility in the Ramsey model, given a rate of time preference ρ and an elasticity of marginal utility θ, we get in equilibrium

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g). \tag{10.60}$$

There will not generally exist a constant, s, such that the right-hand sides of (10.59) and (10.60), respectively, are the same for varying $\tilde{k}$ (that is, outside steady state). But Kurz (1968a) showed the following:

CLAIM Let $\delta, g, n, \alpha,$ and θ be given. If the elasticity of marginal utility θ is greater than 1 and the production function is $\tilde{y} = A\tilde{k}^{\alpha}$ with $\alpha \in (1/\theta, 1)$, then a Ramsey model with $\rho = \theta\alpha(\delta + g + n) - \delta - \theta g$ will generate a constant saving-income ratio $s = 1/\theta$. Thereby the same resource allocation and transitional dynamics arise as in the corresponding Solow model with $s = 1/\theta$.

Proof. Let $1/\theta < \alpha < 1$ and $f(\tilde{k}) = A\tilde{k}^{\alpha}$. Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1}$. The right-hand-side of the Solow equation, (10.59), becomes

$$A\alpha\tilde{k}^{\alpha-1}[s - \frac{(\delta+g+n)\tilde{k}_t}{A\tilde{k}^{\alpha}}] = sA\alpha\tilde{k}^{\alpha-1} - \alpha(\delta+g+n). \tag{10.61}$$

The right-hand-side of the Ramsey equation, (10.60), becomes

$$\frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \frac{\delta+\rho+\theta g}{\theta}.$$

By inserting $\rho = \theta\alpha(\delta+g+n) - \delta - \theta g$, this becomes

$$\begin{aligned} & \frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \frac{\delta+\theta\alpha(\delta+g+n)-\delta-\theta g+\theta g}{\theta} \\ = & \frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \alpha(\delta+g+n). \end{aligned} \tag{10.62}$$

For the chosen ρ we have $\rho = \theta\alpha(\delta+g+n) - \delta - \theta g > n + (1-\theta)g$, because $\theta\alpha > 1$ and $\delta+g+n > 0$. Thus, $\rho - n > (1-\theta)g$ and existence of equilibrium in the Ramsey model with this ρ is ensured. We can now make (10.61) and (10.62) the same by inserting $s = 1/\theta$. This also ensures that the two models require the same $\tilde{k}^*$ to obtain a *constant* $\tilde{c} > 0$. With this $\tilde{k}^*$, the requirement $\dot{\tilde{k}}_t = 0$ gives the same steady-state value of $\tilde{c}$ in both models, in view of (10.58). It follows that $(\tilde{k}_t, \tilde{c}_t)$ is the same in the two models for all $t \geq 0$. □

On the other hand, maintaining $\tilde{y} = A\tilde{k}^{\alpha}$, but allowing $\rho \neq \theta\alpha(\delta+g+n)$ $-\delta - \theta g$, so that $\theta \neq 1/s^*$, then $s'(\tilde{k}) \neq 0$, i.e., the Ramsey model does not generate a constant saving-income ratio except in steady state. Defining s^* as in (10.40) and $\bar{\theta} \equiv (\delta+\rho)/[\alpha(\delta+g+n)-g]$, we have: When $\alpha(\delta+g+n) > g$ (which seems likely empirically), it holds that if $\theta \lesseqgtr 1/s^*$ (i.e., if $\theta \lesseqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqgtr 0$, respectively; if instead $\alpha(\delta+g+n) \leq g$, then $\theta < 1/s^*$ and $s'(\tilde{k}) < 0$, unconditionally. These results follow by considering the slope of the saddle path in a phase diagram in the $(\tilde{k}, \tilde{c}/f(\tilde{k}))$ plane and using that $s(\tilde{k}) = 1 - \tilde{c}/f(\tilde{k})$, cf. Exercise 10.?? The intuition is that when $\tilde{k}$ is rising over time (i.e., society is becoming wealthier), then, when the desire for consumption smoothing is "high" (θ "high"), the prospect of high consumption in the future is partly taken out as high consumption already today, implying that saving is initially low, but rising over time until it eventually settles down in the steady state. But if the desire for consumption smoothing is "low" (θ "low"), saving will initially be high and then gradually fall in the process toward the steady state. The case where $\tilde{k}$ is falling over time gives symmetric results.

E. The social planner's solution

In the text of Section 10.5 we postponed some of the technical details. First, by (A2), the existence of the steady state, E, and the saddle path in Fig. 10.2 is ensured. Solving the linear differential equation (10.46) gives $\lambda_t = \lambda_0 e^{-\int_0^t (f'(\tilde{k}_s)-\delta-\hat{\rho}-g)ds}$. Substituting this into the transversality condition (10.47) gives

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s)-\delta-g-n)ds} = 0, \tag{10.63}$$

where we have eliminated the unimportant positive factor $\lambda_0 = c_0^{-\hat{\theta}} T_0 > 0$.

The condition (10.63) is essentially the same as the transversality condition (10.30) for the market economy and holds in the steady state, given the parameter restriction $\hat{\rho} - n > (1-\hat{\theta})g$, which is analogue to (A1). Thus, (10.63) also holds along the saddle path. Since we must have $\tilde{k} \geq 0$ for all $t \geq 0$, (10.63) has the form required by Mangasarian's sufficiency theorem. If we can show that the Hamiltonian is jointly concave in $(\tilde{k}, c)$ for all $t \geq 0$, then the saddle path *is* a solution to the social planner's problem. And if we can show strict concavity, the saddle path is the *unique* solution. We have:

$$\begin{aligned}
\frac{\partial H}{\partial \tilde{k}} &= \lambda(f'(\tilde{k}) - (\delta + g + n)), \quad \frac{\partial H}{\partial c} = c^{-\hat{\theta}} - \frac{\lambda}{T}, \\
\frac{\partial^2 H}{\partial \tilde{k}^2} &= \lambda f''(\tilde{k}) < 0 \quad (\text{by } \lambda = c^{-\hat{\theta}} T > 0), \quad \frac{\partial^2 H}{\partial c^2} = -\hat{\theta} c^{-\hat{\theta}-1} < 0, \\
\frac{\partial^2 H}{\partial \tilde{k} \partial c} &= 0.
\end{aligned}$$

So the leading principal minors of the Hessian matrix of H are

$$D_1 = -\frac{\partial^2 H}{\partial \tilde{k}^2} > 0, \quad D_2 = \frac{\partial^2 H}{\partial \tilde{k}^2}\frac{\partial^2 H}{\partial c^2} - \left(\frac{\partial^2 H}{\partial \tilde{k} \partial c}\right)^2 > 0.$$

Hence, H is strictly concave in $(\tilde{k}, c)$ and the saddle path is the *unique* optimal solution.

It also follows that the transversality condition (10.47) *is* a necessary optimality condition when the parameter restriction $\hat{\rho} - n > (1-\hat{\theta})g$ holds. Note that we have had to derive this conclusion in a different way than when solving the household's consumption/saving problem in Section 10.2. There we could appeal to a link between the No-Ponzi-Game condition (with strict equality) and the transversality condition to verify necessity of the transversality condition. But that proposition does not cover the social planner's problem where there is no NPG condition.

As to the diverging paths in Fig. 10.2, note that paths of type II (those paths which, as shown in Appendix C, in finite time deplete all capital) can not be optimal, in spite of the temporarily high consumption level. This follows from the fact that the saddle path is the unique solution. Finally, paths of type III in Fig. 10.2 behave as in (10.56) and thus violate the transversality condition (10.47), as claimed in the text.

10.9 Exercises