Chapter 7

Bequests and the modified golden rule

This chapter modifies the Diamond OLG model by including a *bequest* motive. Depending on what form and what strength the bequest motive has, distinctive new conclusions may arise. Indeed, under certain conditions the long-run real interest rate, for instance, turns out to be determined by a very simple principle, the *modified golden rule*. The chapter also spells out both the logic and the limitations of the hypothesis of Ricardian equivalence (government debt neutrality).

7.1 Bequests

In Diamond's OLG model individuals care only about their own lifetime utility and never leave bequests. Yet, in reality a sizeable part of existing private wealth is attributable to inheritance rather than own life-cycle saving. Among empiricists there is considerable disagreement as to the exact quantities, though. Kotlikoff and Summers (1981) estimate that 70-80 % of private financial wealth in the US is attributable to intergenerational transfers and only 20-30 % to own life-cycle savings. On the other hand, Modigliani (1988) suggests that these proportions more or less should be reversed.

The possible motives for leaving bequests include:

1. "Altruism". Parents care about the welfare of their descendants and leave bequests accordingly. This is the hypothesis advocated by the American economist Robert Barro (1974). Its implications are the main theme of this chapter.

2. "Joy of giving". Parents' utility depends not on descendants' utility, as with motive 1, but directly on the size of the bequest. That is, parents

simply have taste for generosity. They enjoy giving presents to their children (Andreoni, 1989, Bossman et al., 2007, Benhabib et al., 2011). Or a more sinister motive may be involved, such as the desire to manipulate your children's behavior (Bernheim et al., 1985).

3. "Joy of wealth". Dissaving may be undesirable even at old age if wealth, or the power and prestige which is associated with wealth, is an independent argument in the utility function (Zou, 1995). Then the profile of individual financial wealth through life may have positive slope at all ages. At death the wealth is simply passed on to the heirs.

In practice an important factor causing bequests is uncertainty about time of death combined with the absence of complete annuity markets. In this situation unintentional bequests arise. Gale and Scholz (1994) find that only about half of net wealth accumulation in the US represents intended transfers and bequests.

How transfers and bequests affect the economy depends on the reasons why they are made. We shall concentrate on a model where bequests reflect the concern of parents for the welfare of their offspring (motive 1 above).

7.2 Barro's dynasty model

We consider a model of overlapping generations linked through altruistic bequests, suggested by Barro (1974). Among the interesting results of the model are that if the extent of altruism is sufficiently high so that the bequest motive is operative in every period, then:

- The differences in age in the population becomes inconsequential; the household sector appears as consisting of a finite number of infinitely-lived dynasties, all alike. In brief, the model becomes a *representative agent model.*
- A simple formula determining the long-run real interest rate arises: the *modified golden rule.*
- *Ricardian equivalence* arises.
- Resource allocation in a competitive market economy coincides with that accomplished by a social planner who has the same intergenerational discount rate as the representative household dynasty.

This chapter considers the three first bullets in detail, while the last bullet is postponed to the next chapter.

7.2.1 A forward-looking altruistic parent

Technology, demography, and market conditions are as in Diamond's OLG model. There is no utility from leisure. Perfect foresight is assumed. Until further notice technological progress is ignored.

The preferences of a member of generation t are given by the utility function

$$U_t = u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1}) + (1+R)^{-1}(1+n)U_{t+1}. \tag{7.1}$$

Here R is the pure *inter*generational utility discount rate and $1+n$ is the number of offspring per parent in society. So R measures the extent of "own generation preference" (strength of "self preference") and n is the population growth rate. As usual we in (7.1) ignore indivisibility problems and take an average view. The term $u(c_{1t})$ $+(1+\rho)^{-1}u(c_{2t+1})$ is the "own lifetime utility", reflecting the utility contribution from one's own consumption as young, c_{1t}, and as old, c_{2t+1}.The time preference parameter ρ appears as the *intra*generational utility discount rate. Although $\rho > 0$ is plausible, the results to be derived do not depend on this inequality, so we just impose the formal restriction that $\rho > -1$.

The term $(1+R)^{-1}(1+n)U_{t+1}$ in (7.1) is the contribution derived from the utility of the offspring. The intergenerational utility discount factor $(1+R)^{-1}$ is also known as the *altruism* factor.

The *effective* intergenerational utility discount rate is the number $\bar{R}$ satisfying

$$(1+\bar{R})^{-1} = (1+R)^{-1}(1+n). \tag{7.2}$$

We assume $R > n \geq 0$. So $\bar{R}$ is positive and the utility of the next generation is weighed through an *effective* discount factor $(1+\bar{R})^{-1} < 1$. (Mathematically, the model works well with the weaker assumption, $n > -1$; yet it helps intuition to imagine that there always is at least one child per parent.)

We write (7.1) on recursive form,

$$U_t = V_t + (1+\bar{R})^{-1}U_{t+1},$$

where V_t is the "direct utility" $u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1})$, whereas $(1+\bar{R})^{-1}U_{t+1}$ is the "indirect utility" through the offspring's well-being. By forward substitution j periods ahead in (7.1) we get

$$U_t = \sum_{i=0}^{j}(1+\bar{R})^{-i}V_{t+i} \quad + \quad (1+\bar{R})^{-(j+1)}U_{t+j+1}.$$

Provided U_{t+j+1} does not grow "too fast", taking the limit for $j \to \infty$ gives

$$U_t = \sum_{i=0}^{\infty}(1+\bar{R})^{-i}V_{t+i} = \sum_{i=0}^{\infty}(1+\bar{R})^{-i}\left[u(c_{1t+i}) + (1+\rho)^{-1}u(c_{2t+i+1})\right]. \tag{7.3}$$

By $\sum_{i=0}^{\infty}(1+\bar{R})^{-i}V_{t+i}$ we mean $\lim_{j\to\infty}\sum_{i=0}^{j}(1+\bar{R})^{-i}V_{t+i}$, assuming this limit exists. Although each generation cares directly only about the next generation, this series of intergenerational links implies that each generation acts as if it cared about all future generations in the dynastic family, although with decreasing weight. In this way the entire dynasty can be regarded as a single agent with infinite horizon. At the same time the coexisting dynasties are completely alike. So, in spite of starting from an OLG structure, we now have a *representative agent model*, i.e., a model where the household sector consists of completely alike decision makers.

Note that *one-parent* families fit Barro's notion of clearly demarcated dynastic families best. Indeed, the model abstracts from the well-established fact that breeding arises through mating between a man and a woman who come from two *different* parent families. In reality this gives rise to a complex network of interconnected families. Until further notice we ignore such complexities and proceed by imagining reproduction is not sexual.

In each period two (adult) generations coexist: the "young", each of whom supplies one unit of labor inelastically, and the "old" who do not supply labor. Each old is a parent to $1+n$ of the young. And each young is a parent to $1+n$ children, born when the young parent enters the economy and the grandparent retires from the labor market. Next period these children become visible in the model as the young generation in the period.

Let b_t be the bequest received at the end of the first period of "economic life" by a member of generation t from the old parent, belonging to generation $t-1$. In turn this member of generation t leaves in the next period a bequest to the next generation in the family and so on. We will assume, realistically, that negative bequests are ruled out by law; the legal system exempts children from responsibility for parental debts. Thus the budget constraints faced by a young member of generation t are:

$$c_{1t}+s_t = w_t+b_t, \tag{7.4}$$

$$c_{2t+1}+(1+n)b_{t+1} = (1+r_{t+1})s_t, \qquad b_{t+1}\geq 0, \tag{7.5}$$

where s_t denotes saving as young (during work life) out of the sum of labor income and the bequest received (payment for the consumption and receipt of w_t+b_t occur at the end of the period). In the next period the person is an old parent and ends life leaving a bequest, b_{t+1}, to each of the $1+n$ children. Fig. 7.1 illustrates.

What complicates the analysis is that even though a bequest motive is present, the market circumstances may be such that parents do not find it worthwhile to transmit positive bequests. We then have a corner solution, $b_{t+1}=0$. An important element in the analysis is to establish *when* this occurs and when it

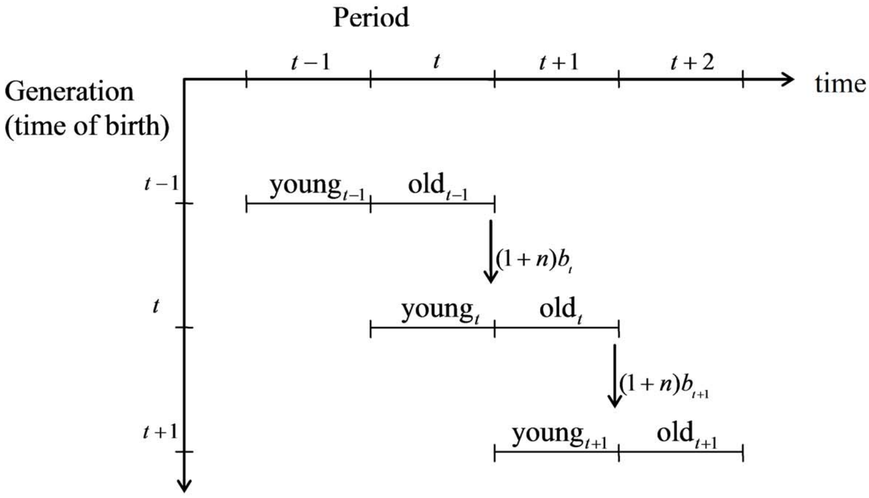

Figure 7.1: The generational structure of the Barro model.

does not. To rule out the theoretical possibility of a corner solution also for s_t, we impose the No Fast Assumption

$$\lim_{c\to 0} u'(c) = \infty. \tag{A1}$$

Given this assumption, the young will always choose $s_t > 0$ as soon as $w_t + b_t > 0$.

Consider a person belonging to generation t. This person has perfect foresight with regard to future wages and interest rates and can compute the optimal choices of the descendants conditional on the bequest they receive. The planning problem is:

$$\begin{aligned} \max U_t &= u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1}) \\ &\quad +(1+\bar{R})^{-1}\left[u(c_{1t+1}) + (1+\rho)^{-1}u(c_{2t+2})\right] + \ ... \end{aligned}$$

subject to the budget constraints (7.4) and (7.5) and knowing that the descendants will respond optimally to the received bequest (see below). We insert into U_t the two budget constraints in order to consider the objective of the parent as a function, $\tilde{U}_t$, of the decision variables, s_t and b_{t+1}. We then maximize with respect to s_t and b_{t+1}. First:

$$\begin{aligned} \frac{\partial \tilde{U}_t}{\partial s_t} &= -u'(c_{1t}) + (1+\rho)^{-1}u'(c_{2t+1})(1+r_{t+1}) = 0, \text{ i.e.,} \\ u'(c_{1t}) &= (1+\rho)^{-1}u'(c_{2t+1})(1+r_{t+1}). \end{aligned} \tag{7.6}$$

This first-order condition deals with the distribution of own consumption across time. The condition says that in the optimal plan the opportunity cost of saving one more unit as young must equal the discounted utility benefit of having $1+r_{t+1}$ more units for consumption as old.

As to maximizing with respect to the second decision variable, b_{t+1}, we have to distinguish between the case where the bequest motive is operative and the case where it is not. Here we consider the first case and postpone the second for a while.

7.2.2 Case 1: the bequest motive operative ($b_{t+1} > 0$ optimal)

In addition to (7.6) we get the first-order condition

$$\frac{\partial \tilde{U}_t}{\partial b_{t+1}} = (1+\rho)^{-1}u'(c_{2t+1})\left[-(1+n)\right] + (1+\bar{R})^{-1}u'(c_{1t+1}) \cdot 1 = 0, \text{ i.e.,}$$

$$(1+\rho)^{-1}u'(c_{2t+1}) = (1+\bar{R})^{-1}u'(c_{1t+1})(1+n)^{-1}. \tag{7.7}$$

This first-order condition deals with the distribution of consumption across generations in the same period. The condition says that in the optimal plan the parent's utility cost of increasing the bequest by one unit (thereby decreasing the consumption as old by one unit) must equal the discounted utility benefit derived from the next generation having $1/(1+n)$ more units, per member, for consumption in the same period. The factor, $(1+n)^{-1}$, is a "dilution factor" due to total bequests being diluted in view of the $1+n$ children for each parent.

A further necessary condition for an optimal plan is that the bequests are not forever too high, which would imply postponement of consumption possibilities forever. That is, we impose the condition

$$\lim_{i\to\infty}(1+\bar{R})^{-(i-1)}(1+\rho)^{-1}u'(c_{2t+i})(1+n)b_{t+i} = 0. \tag{7.8}$$

Such a terminal condition is called a *transversality condition*; it acts as a necessary first-order condition at the terminal date, here at infinity. Imagine a plan where instead of (7.8) we had $\lim_{i\to\infty}(1+\bar{R})^{-(i-1)}(1+\rho)^{-1}u'(c_{2t+i})(1+n)b_{t+i} > 0$. In this case there would be "over-bequeathing" in the sense that U_t (the sum of the generations' discounted lifetime utilities) could be increased by the ultimate generation consuming more as old and bequeathing less. Decreasing the ultimate bequest to the young, b_{t+i} ($i \to \infty$), by one unit would imply $1+n$ extra units for consumption for the old parent, thereby increasing this parent's utility by $(1+\rho)^{-1}u'(c_{2t+i})(1+n)$. From the perspective of the current generation t this utility contribution should be discounted by the discount factor $(1+\bar{R})^{-(i-1)}$. With a finite time horizon entailing that only $i-1$ future generations (i fixed) were cared about, it would be waste to end up with $b_{t+i} > 0$; optimality would require $b_{t+i} = 0$. The condition (7.8) is an extension of this principle to an infinite horizon.[1]

The optimality conditions (7.6), (7.7), and (7.8) illustrate a general principle of intertemporal optimization. First, no gain should be achievable by a reallocation of resources between two periods or between two generations. This is taken care of by the *Euler equations* (7.6) and (7.7).[2] Second, there should be nothing of value leftover after the "last period", whether the horizon is finite or infinite. This is taken care of by a transversality condition, here (7.8). With a finite horizon, the transversality condition takes the simple form of a condition saying that the intertemporal budget constraint is satisfied with equality. In the two-period models of the preceding chapters, transversality conditions were implicitly satisfied in that the budget constraints were written with = instead of $\leq$.

The reader might be concerned whether in our maximization procedure, in particular regarding the first-order condition (7.7), we have taken the descendants' optimal responses properly into account. The parent should choose s_t and b_{t+1} to maximize U_t, taking into account the descendants' optimal responses to the received bequest, b_{t+1}. In this perspective it might seem inadequate that we have considered only the partial derivative of U_t w.r.t. b_{t+1}, not the total derivative. Fortunately, in view of the *envelope theorem* our procedure is valid. Applied to the present problem, the envelope theorem says that in an interior optimum the total derivative of U_t w.r.t. b_{t+1} *equals* the partial derivative w.r.t. b_{t+1}, evaluated at the optimal choice by the descendants. Indeed, since an objective function "is flat at the top", the descendants' response to a marginal change in the received bequest has a negligible effect on the value of optimized objective function (for details, see Appendix A).

The old generation in period t So far we have treated the bequest, b_t, received by the young in the current period, t, as given. But in a sense also this bequest is a choice, namely a choice made by the old parent in this period, hence endogenous. This old parent enters period t with assets equal to the saving made as young, s_{t-1}, which, if period t is the initial period of the model, is part of the arbitrarily *given* initial conditions of the model. From this perspective the decision problem for this old parent is to choose $b_t \geq 0$ so as to

$$\max \left[(1+\rho)^{-1}u(c_{2t}) + (1+\bar{R})^{-1}U_t\right]$$

[1] Although such a simple extension of a transversality condition from a finite horizon to an infinite horizon is not always valid, it *is* justifiable in the present case. This and related results about transversality conditions are dealt with in detail in the next chapter.

[2] Since $\tilde{U}_t(s_t, b_{t+1})$ is jointly strictly concave in its two arguments, the Euler equations are not only necessary, but also sufficient for a unique interior maximum.

subject to the budget constraint $c_{2t} + (1+n)b_t = (1+r_t)s_{t-1}$ and taking into account that the chosen b_t indirectly affects the maximum lifetime utility to be achieved by the next generation. If the optimal b_t is positive, the choice satisfies the first-order condition (7.7) with t replaced by $t-1$. And if no disturbance of the economy has taken place at the transition from period $t-1$ to period t, the decision by the old in period 0 is simply to do exactly as planned when young in the previous period.

It may seem puzzling that $u(c_{2t})$ is discounted by the factor $(1+\rho)^{-1}$, when standing *in* period t. Truly, when thinking of the old parent as maximizing $(1+\rho)^{-1}u(c_{2t}) + (1+\bar{R})^{-1}U_t$, the utility is discounted back (as usual) to the first period of adult life, in this case period $t-1$. But this is just one way of presenting the decision problem of the old. An alternative way is to let the old maximize the present value of utility as seen from the current period t,

$$\begin{aligned}(1+\rho)\left[(1+\rho)^{-1}u(c_{2t}) + (1+\bar{R})^{-1}U_t\right] &= u(c_{2t}) + (1+\bar{R})^{-1}(1+\rho)U_t \\ &\equiv u(c_{2t}) + (1+\psi)^{-1}U_t,\end{aligned}$$

where the last equality follows by merging the backward and forward discounts, $(1+\bar{R})^{-1}$ and $1+\rho$, respectively, into the coefficient $(1+\psi)^{-1}$. Since both utilities, $u(c_{2t})$ and U_t, arise in the *same* period, the coefficient $(1+\psi)^{-1}$ is no time discount factor but an expression for the degree of unselfishness, see the remark below. As we have just multiplied the objective function by a positive constant, $1+\beta$, the resulting behavior is unaffected.

Remark The effective intergenerational utility discount factor can be decomposed as in (7.2) above, but also as:

$$(1+\bar{R})^{-1} \equiv (1+\rho)^{-1}(1+\psi)^{-1}. \tag{7.9}$$

The sub-discount factor, $(1+\rho)^{-1}$, applies because the prospective utility arrives one period later and is, in this respect, comparable to utility from own consumption when old, c_{2t+1}. The sub-discount factor, $(1+\psi)^{-1}$, can be seen as the *degree of unselfishness* and ψ as reflecting the extent of *selfishness*. Indeed, when ψ is positive, parents are selfish in the sense that, if a parent's consumption when old equals the children's' consumption when young, then the parent prefers to keep an additional unit of consumption for herself instead of handing it over to the next generation (replace $(1+\bar{R})^{-1}$ in (7.7) by (7.9)). □

The equilibrium path

Inserting (7.7) on the right-hand side of (7.6) gives

$$u'(c_{1t}) = (1+\bar{R})^{-1}u'(c_{1t+1})\frac{1+r_{t+1}}{1+n}. \tag{7.10}$$

This is an Euler equation characterizing the optimal distribution of consumption across generations in different periods: in an optimal intertemporal and intergenerational allocation, the utility cost of decreasing consumption of the young in period t (that is, saving and investing one more unit) must equal the discounted utility gain next period for the next generation which, per member, will be able to consume $(1 + r_{t+1})/(1 + n)$ more units.

With perfect competition and neoclassical CRS technology, in equilibrium the real wage and interest rate are

$$w_t = f(k_t) - f'(k_t)k_t \text{ and } r_t = f'(k_t) - \delta, \tag{7.11}$$

respectively, where f is the production function on intensive form, and δ is the capital depreciation rate, $0 \leq \delta \leq 1$. Further, $k_t \equiv K_t/L_t$, where K_t is aggregate capital in period t owned by that period's old, and L_t is aggregate labor supply in period t which is the same as the number of young in that period.

As in the Diamond model, aggregate consumption per unit of labor satisfies the technical feasibility constraint

$$\begin{aligned} c_t &\equiv \frac{C_t}{L_t} \equiv (c_{1t}L_t + c_{2t}L_{t-1})/L_t = c_{1t} + c_{2t}/(1+n) \\ &= f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}. \end{aligned} \tag{7.12}$$

An *equilibrium path* for $t = 0, 1, 2, \ldots$, is described by the first-order conditions (7.7) ("backwarded" one period) and (7.10), the transversality condition (7.8), the equilibrium factor prices in (7.11), the resource constraint (7.12), and an initial condition in the form of a given $k_0 > 0$. This k_0 may be interpreted as reflecting a given $s_{-1} \geq 0$. Indeed, as in the Diamond model, for every $t = 0, 1, \ldots$, we have

$$k_{t+1} \equiv \frac{K_{t+1}}{L_{t+1}} = \frac{s_t L_t}{L_{t+1}} = \frac{s_t}{1+n}. \tag{7.13}$$

This is simply a matter of accounting. At the beginning of period $t+1$ the available aggregate capital stock equals the financial wealth of the generation which now is old but was young in the previous period and saved $s_t L_t$ out of that period's labor income plus received transfers in the form of bequests. So $K_{t+1} = s_t L_t$. Indeed, the new young generation of period $t + 1$ own to begin with nothing except their brain and bare hands, although they *expect* to receive a bequest just before they retire from the labor market at the end of period $t+1$. Still another interpretation of (7.13) starts from the general accounting principle for a closed economy that the increase in the capital stock equals aggregate net saving. In turn aggregate net saving is the sum of net saving by the young, S_{1t}, and net saving by the old, S_{2t}:

$$\begin{aligned} K_{t+1} - K_t &= S_{1t} + S_{2t} = s_t L_t + \left[r_t s_{t-1} - (c_{2t} + (1+n)b_t) \right] L_{t-1} \\ &= s_t L_t + \left[r_t s_{t-1} L_{t-1} - (1+r_t) s_{t-1} L_{t-1} \right] \qquad \text{(by ((7.5))} \\ &= s_t L_t + (-s_{t-1} L_{t-1}) = s_t L_t - K_t, \end{aligned} \tag{7.14}$$

which by eliminating $-K_t$ on both sides gives $K_{t+1} = s_t L_t$.

One of the results formally proved in the next chapter is that, given the technology assumption $\lim_{k\to 0} f'(k) > R + \delta > \lim_{k\to\infty} f'(k)$, an equilibrium path exists and converges to a steady state. In that chapter it is also shown that the conditions listed are exactly those that also describe a certain "command optimum". This refers to the allocation brought about by a social planner having (7.3) as the social welfare function.

Steady state with an operative bequest motive: the modified golden rule

A steady state of the system is a state where, for all t, $k_t = k^*$, $c_t = c^*$, $c_{1t} = c_1^*$, $c_{2t} = c_2^*$, $b_t = b^*$, and $s_t = s^*$. In a steady state with $b^* > 0$, we have the remarkably simple result that the interest rate, r^*, satisfies the *modified golden rule*:

$$1 + r^* = 1 + f'(k^*) - \delta = (1 + \bar{R})(1 + n) \equiv 1 + R. \tag{7.15}$$

This follows from inserting the steady state conditions ($k_t = k^*$, $c_{1t} = c_1{}^*$, $c_{2t} = c_2{}^*$, for all t) into (7.10) and rearranging. In the *golden rule* of Chapter 3 the interest rate (reflecting the net marginal productivity of capital) equals the output growth rate (here n). The "modification" here comes about because of the strictly positive effective intergenerational discount rate $\bar{R}$, which implies a higher interest rate.

Two things are needed for the economy with overlapping generations linked through bequests to be in a steady state with positive bequests. First, it is necessary that the rate of return on saving matches the rate of return, R, *required* to tolerate a marginal decrease in own current consumption for the benefit of the next generation. This is what (7.10) shows. Second, it is necessary that the rate of return induces an amount of saving such that the consumption of each of the children equals the parent's consumption as young in the previous period. Otherwise the system would not be in a steady state. If in (7.15) "=" is replaced by ">" ("<"), then there would be a temptation to save more (less), thus generating more (less) capital accumulation, thereby pushing the system away from a steady state.

A steady state with an operative bequest motive is unique. Indeed, (7.15) determines a unique k^* (since $f'' < 0$), which gives $s^* = (1 + n)k^*$ by (7.13)

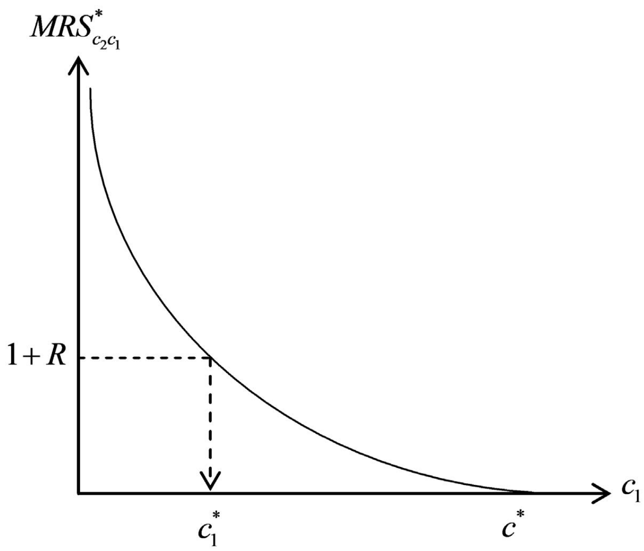

Figure 7.2: Steady state consumption as young.

and determines c^* uniquely from the second line in (7.12). In turn, we can find c_1^* and c_2^* from the Euler equation (7.6) which implies that the marginal rate of substitution of c_2 for c_1 in steady state takes the form

$$MRS^*_{c_2c_1} = \frac{u'(c_1^*)}{(1+\rho)^{-1}u'(c_2^*)} = \frac{u'(c_1^*)}{(1+\rho)^{-1}u'\left((1+n)(c^* - c_1^*)\right)} = 1 + r^* = 1 + R, \tag{7.16}$$

where the second equality comes from (7.12). Given c^*, $MRS_{c_2c_1}$ is a decreasing function of c_1^* so that a solution for c_1^* in (7.16) is unique. In view of the No Fast Assumption (A1), (7.16) always *has* a solution in c_1^*, cf. Fig. 7.2. Then, from (7.16), $c_2^* = (1+n)(c^* - c_1^*)$. Finally, from the period budget constraint (7.4), $b^* = c_1^* + s^* - w^* = c_1^* + (1+n)k^* - w^*$, where $w^* = f(k^*) - f'(k^*)k^*$, from (7.11).

But what if the market circumstances and preferences in combination are such that the constraint $b_{t+1} \geq 0$ becomes binding? Then the bequest motive is not operative. We get a corner solution such that the equality sign in (7.15) is replaced by $\leq$. The economy behaves like Diamond's OLG model and a steady state of the economy is not necessarily unique. To see these features, we now reconsider the young parent's optimization problem.

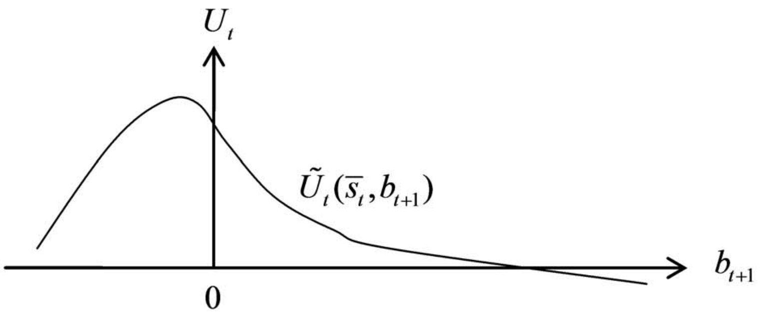

Figure 7.3: Configuration where the constraint $b_{t+1} \geq 0$ is binding (s_t fixed at $\bar{s}_t$).

7.2.3 Case 2: the bequest motive not operative ($b_{t+1} = 0$ optimal)

The first-order condition (7.6) involving s_t is still valid. But the first-order condition involving b_{t+1} becomes an inequality:

$$\frac{\partial \tilde{U}_t}{\partial b_{t+1}} = (1+\rho)^{-1} u'(c_{2t+1}) \left[-(1+n)\right] + (1+\bar{R})^{-1} u'(c_{1t+1}) \cdot 1 \leq 0, \quad \text{i.e.,}$$

$$(1+\rho)^{-1} u'(c_{2t+1}) \geq (1+\bar{R})^{-1} u'(c_{1t+1})(1+n)^{-1}. \tag{7.17}$$

This condition says that in an optimal plan the parent's utility cost of increasing the bequest by one unit of account as an old parent must be larger than or equal to the discounted benefit derived from the next generation having $(1+n)^{-1}$ more units for consumption in the same period.

Why can we not exclude the possibility that $\partial \tilde{U}_t / \partial b_{t+1} < 0$ in the first line of (7.17) when $b_{t+1} = 0$ is optimal? To provide an answer, observe first that if we had $\partial \tilde{U}_t / \partial b_{t+1} > 0$ for $b_{t+1} = 0$, then at the prevailing market conditions a state with $b_{t+1} = 0$ could not be an optimum for the individual. Instead positive bequests would be induced. If, however, for $b_{t+1} = 0$, we have $\partial \tilde{U}_t / \partial b_{t+1} \leq 0$, then at the prevailing market conditions the state $b_{t+1} = 0$ *is* optimal for the individual. This is so even if "<" holds, as in Fig. 7.3. Although the market circumstances here imply a *temptation* to decrease b_{t+1} from the present zero level, by law that temptation cannot be realized.

Substituting (7.17) and $r_{t+1} = f'(k_{t+1}) - \delta$ into (7.6) implies that (7.10) is replaced by

$$u'(c_{1t}) \geq (1+\bar{R})^{-1} u'(c_{1t+1}) \frac{1 + f'(k_{t+1}) - \delta}{1+n}. \tag{7.18}$$

Inserting into this the steady state conditions ($k_t = k^*$, $c_{1t} = c_1{}^*$, $c_{2t} = c_2{}^*$, for all t) and rearranging give

$$1 + r^* = 1 + f'(k^*) - \delta \leq (1+\bar{R})(1+n) \equiv 1 + R. \tag{7.19}$$

Since optimal b_{t+1} in case 2 is zero, everything is as if the bequest-motivating term, $(1+R)^{-1}(1+n)U_{t+1}$, were eliminated from the right-hand side of (7.1). Thus, as expected the model behaves like a Diamond OLG model.

7.2.4 Necessary and sufficient conditions for the bequest motive to be operative

An important question is: under what conditions does the bequest motive turn out to be operative? To answer this we limit ourselves to an analysis of a neighborhood of the steady state. We consider the thought experiment: if the bequest term, $(1+\bar{R})^{-1}U_{t+1}$, is eliminated from the right-hand side of (7.1), what will the interest rate be in a steady state? The utility function then becomes

$$\bar{U}_t = u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1}),$$

which is the standard lifetime utility function in a Diamond OLG model. We will call $\bar{U}_t$ the *truncated utility function* associated with the given true utility function, U_t. The model resulting from replacing the true utility function of the economy by the truncated utility function will be called the *associated Diamond economy.*

It is convenient to assume that the associated Diamond economy is well-behaved, in the sense of having a unique non-trivial steady state. Let r_D denote the interest rate in this Diamond steady state (the suffix D for Diamond). Then:

PROPOSITION 1 *(the cut-off value for the own-generation preference, R)* Consider an economy with a bequest motive as in (7.1) and satisfying the No Fast Assumption (A1). Let $R > n$, i.e., $\bar{R} > 0$. Suppose that the associated Diamond economy is well-behaved and has steady-state interest rate r_D. Then in a steady state of the economy with a bequest motive, bequests are positive if and only if

$$R < r_D. \tag{*}$$

Proof. From (7.7) we have in steady state of the original economy with a bequest motive:

$$\begin{aligned}
\frac{\partial \tilde{U}_t}{\partial b_{t+1}} &= (1+\rho)^{-1}u'(c_2{}^*)(-(1+n)) + (1+\bar{R})^{-1}u'(c_1{}^*) \\
&= -(1+\rho)^{-1}(1+n)u'(c_2{}^*) \\
&\quad +(1+\bar{R})^{-1}(1+\rho)^{-1}(1+r^*)u'(c_2{}^*) \qquad \text{(from (7.6))} \\
&= (1+\rho)^{-1}u'(c_2{}^*)\left[(1+\bar{R})^{-1}(1+r^*) - (1+n)\right] \gtreqless 0 \text{ if and only if} \\
1+r^* &\gtreqless (1+\bar{R})(1+n), \text{ i.e., if and only if } r^* \gtreqless R, \text{ respectively,} \qquad (**)
\end{aligned}$$

since $(1+\bar{R})(1+n) \equiv 1+R$.

The "if" part: Suppose the associated Diamond economy is in steady state with $r_D > R$. Can this Diamond steady state (where by construction $b_{t+1} = 0$) be an equilibrium also for the original economy? The answer is no because, if we assume it were an equilibrium, then the interest rate would be $r^* = r_D > R$ and, by (**), $\partial \tilde{U}_t / \partial b_{t+1} > 0$. Therefore the parents would raise b_{t+1} from its hypothetical zero level to some positive level, contradicting the assumption that $b_{t+1} = 0$ were an equilibrium.

The "only if" part: Suppose instead that $r_D < R$. Can this Diamond steady state (where again $b_{t+1} = 0$, of course) be an equilibrium also for the original economy with a bequest motive? Yes. From (**) we have $\partial \tilde{U}_t / \partial b_{t+1} < 0$. Therefore the parents would be tempted to decrease their b_{t+1} from its present zero level to some negative level, but that is not allowed. Hence, $b_{t+1} = 0$ is still an equilibrium and the bequest motive does not become operative. Similarly, in a case where $R = r_D$, the situation $b_{t+1} = 0$ is still an equilibrium of the economy with a bequest motive, since we get $\partial \tilde{U}_t / \partial b_{t+1} = 0$ when $b_{t+1} = 0$. □

Thus bequests will be positive if and only if the own-generation preference, R, is sufficiently small or, what amounts to the same, the altruism factor, $(1+R)^{-1}$, is sufficiently large − in short, if and only if parents "love their children enough". Fig. 7.4, where k_{MGR} is defined by $f'(k_{MGR}) - \delta = R$, gives an illustration. If the rate R at which the parent discounts the utility of the next generation is relatively high, then k_{MGR} is relatively low and (*) will not be satisfied. This is the situation depicted in the upper panel of Fig. 7.4 (low altruism). In this case the bequest motive will not be operative and the economy ends up in the Diamond steady state with $k^* = k_D > k_{MGR}$. If on the other hand the own-generation preference, R, is relatively low as in the lower panel (high altruism), then (*) is satisfied, the bequest motive will be operative and motivates more saving so that the economy ends up in a steady state satisfying the modified golden rule. That is, (7.15) holds and $k^* = k_{MGR}$.

Both cases portrayed in Fig. 7.4 have $k_D < k_{GR}$, where k_{GR} is the golden rule capital-labor ratio satisfying $f'(k_{GR}) - \delta = n$. But theoretically, we could equally well have $k_D > k_{GR}$ so that the Diamond economy would be dynamically inefficient. The question arises: does the presence of a bequest motive help to eliminate the potential for dynamic inefficiency? The answer is given by point (i) of the following corollary of Proposition 1.

COROLLARY Let $R > n$. The economy with a bequest motive is:
(i) dynamically inefficient if and only if the associated Diamond economy is dynamically inefficient; and
(ii) the economy has positive bequests in steady state only if it is dynamically

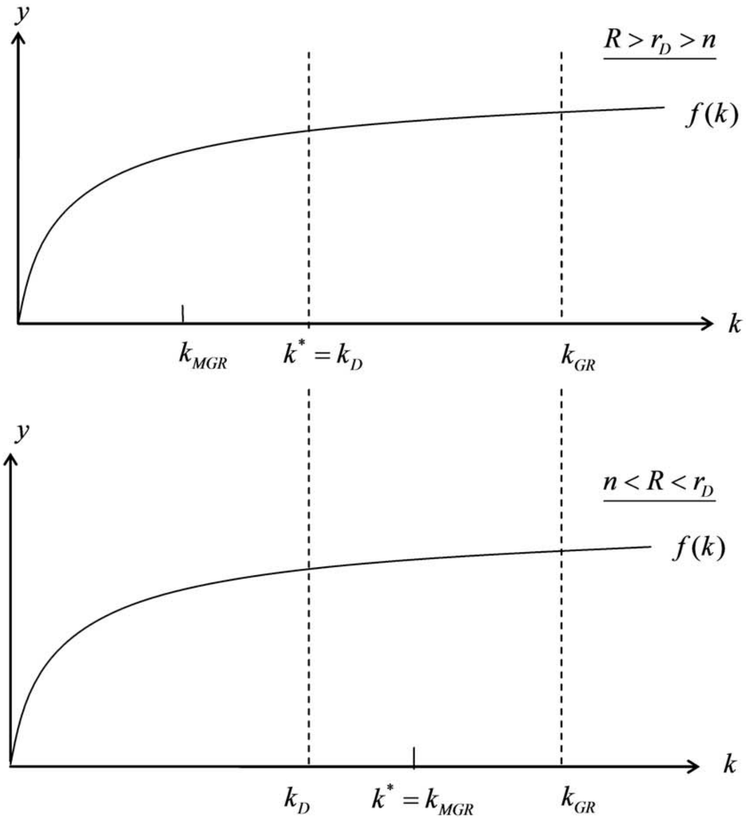

Figure 7.4: How the steady-state capital-labor ratio depends on the size of the own-generation preference R (for a given $k_D < k_{GR}$). Upper panel: high R results in zero bequest. Lower panel: low R induces additional saving and positive bequest so that k_{MGR}, which is larger the lower R is, becomes a steady state instead of k_D (k_{MGR} satisfies $f'(k_{MGR}) - \delta = R > n$ and k_{GR} satisfies $f'(k_{GR} - \delta = n)$.

efficient.

Proof. (i) "if": suppose $r_D < n$. Then, since by assumption $R > n$, $(*)$ is not satisfied. Hence, the bequest motive cannot be operative and the economy behaves like the associated Diamond economy which is dynamically inefficient in view of $r_D < n$.

(i) "only if": suppose $r^* < n$. Then the economy with a bequest motive is dynamically inefficient. Since by assumption $R > n$, it follows that $r^* < R$. So (7.15) is not satisfied, implying by Proposition 1 that bequests cannot be positive. Hence, the allocation is as in the associated Diamond economy which must then also be dynamically inefficient.

(ii) We have just shown that $r^* < n$ implies zero bequests. Hence, if there are positive bequests, we must have $r^* \geq n$, implying that the economy is dynamically efficient. □

The corollary shows that the presence of a bequest motive does not help in eliminating a tendency for dynamic inefficiency. This is not surprising. Dynamic inefficiency arises from perpetual excess saving. A bequest motive cannot be an incentive to save less (unless negative bequests are allowed). On the contrary, when a bequest motive is operative, you are motivated to increase saving as young in order to leave bequests; aggregate saving will be higher. That is why, when $R < r_D$, the economy ends up, through capital accumulation, in a steady state with $r^* = R$, so that r^* is smaller than r_D (though $r^* > n$ still). On the other hand, by the corollary follows also that the bequest motive can only be operative in a dynamically efficient economy. Indeed, we saw that an operative bequest motive implies the modified golden rule (7.15) so that $r^* = R$ where, by assumption, $R > n$ and therefore $k_{MGR} < k_{GR}$ always. (If $R \leq n$, the effective utility discount factor, $(1 + \bar{R})^{-1}$, in (7.3) is no longer less than one, which may result in non-existence of general equilibrium.)

7.3 Bequests and Ricardian equivalence

As we have seen, when the bequest motive is operative, the Barro model becomes essentially a *representative agent* model in spite of starting from an OLG structure. So aggregate household behavior is simply a multiple of the behavior of a single dynasty.

This feature has implications for the issue of Ricardian equivalence.[3] To see this, we add a government sector to the model. We assume that the government

[3] A first discussion of this issue, based on a different model, the Diamond OLG model, appears at the end of Chapter 6.

finances its spending sometimes by lump-sum taxation, sometimes by issuing debt. *Ricardian equivalence*, also called *debt neutrality*, is then present if, for a given path of government spending, a shift between tax and debt financing does not affect resource allocation. That is, a situation with a tax cut and ensuing budget deficit is claimed to be "*equivalent*" to a situation without the tax cut.

Barro (1974) used the above model to substantiate this claim. Faced with a tax cut in period t, the current generations will anticipate higher taxes in the future. Indeed, to cover the government's higher future interest payment, the present value of future taxes will have to rise exactly as much as current taxes are decreased. Assuming the government waits j periods to increase taxes and then does it fully once for all in period $t + j$, for each unit of account current taxes are reduced, taxes j periods ahead are increased by $(1+r)^j$ units of account. The present value as seen from the end of period t of this future tax increase is $(1 + r)^j/(1 + r)^j = 1$. So the change in the time profile of taxation will neither make the dynasties feel richer or poorer. Consequently, their current and planned future consumption will be unaffected.

The Ricardian Equivalence view is then that to compensate the descendants j generations ahead for the higher taxes, the old generation will save the rise in current after-tax income and leave higher bequests to their descendants (presupposing the bequest motive is operative). And the young generation will increase their saving by as much as *their* after-tax income is raised as a consequence of the tax cut and the higher bequests they expect to receive when retiring. In this way all private agents maintain the consumption level they would have had in the absence of the tax cut. The change in fiscal policy is thus completely nullified by the response of the private sector. The decrease in public saving is offset by an equal increase in aggregate private saving. So national saving as well as consumption remain unaffected.

We now formalize this story, taking population growth and fully specified budget constraints into account. Let

G_t = real government spending on goods and services in period t,

T_t = real tax revenue in period t,

τ_t = T_t/L_t = a lump-sum tax levied on each young in period t,

B_t = real government debt as inherited from the end of period $t - 1$.

To fix ideas, suppose G_t is primarily eldercare, including health services, and therefore proportional to the number of old, i.e.,

$$G_t = \gamma L_{t-1}, \qquad \gamma > 0. \tag{7.20}$$

We assume the public service enter in a separable way in the lifetime utility function so that marginal utilities of private consumption are not affected by G_t.

The resource constraint of the economy now is

$$\begin{aligned} c_t &\equiv \frac{C_t}{L_t} \equiv (c_{1t}L_t + c_{2t}L_{t-1})/L_t = c_{1t} + c_{2t}/(1+n) \\ &= f(k_t) - (1+n)^{-1}\gamma + (1-\delta)k_t - (1+n)k_{t+1}. \end{aligned}$$

instead of (7.12) above.

From time to time the government runs a budget deficit (or surplus) and in such cases, the deficit is financed by bond issue (or withdrawal).[4] Along with interest payments on government debt, elder care is the only government expense. That is,

$$B_{t+1} - B_t = rB_t + G_t - T_t, \qquad B_0 \text{ given}, \tag{7.21}$$

where the real interest rate r is for simplicity assumed constant. We further assume $r > n$; this is in accordance with the above result that when a Barro economy is in steady state with positive bequests, then, ignoring technological progress, by (7.15), the interest rate equals the intergenerational utility discount rate R (to ensure existence of general equilibrium, R was in connection with (7.2) assumed larger than n).

In absence of technological progress the steady state growth rate of the economy equals the growth rate of the labor force, that is, $g_Y = n < r$. Since the interest rate is thus higher than the growth rate, to maintain solvency the government must satisfy its intertemporal budget constraint,

$$\sum_{i=0}^{\infty} G_{t+i}(1+r)^{-(i+1)} \leq \sum_{i=0}^{\infty} T_{t+i}(1+r)^{-(i+1)} - B_t. \tag{7.22}$$

This says that the present value of current and expected future government spending is constrained by government wealth (the present value of current and expected future tax revenue minus existing government debt).

Let us concentrate on the "regular" case where the government does not tax more heavily than needed to cover the spending G_t and the debt service, that is, the government does not want to accumulate financial net wealth. Then there is strict equality in (7.22). Applying (7.20) and that $L_{t+i} = L_t(1+n)^i$ and $T_{t+i} = \tau_{t+i}L_{t+i}$, (7.22) with strict equality simplifies to

$$L_t \sum_{i=0}^{\infty} \frac{(1+n)^i}{(1+r)^{i+1}}(\tau_{t+i} - \frac{\gamma}{1+n}) = B_t. \tag{7.23}$$

Suppose that, for some periods, taxes are cut so that $T_{t+i} < G_{t+i} + rB_{t+i}$, that is, a budget deficit is run. Is resource allocation − aggregate consumption

[4] The model ignores money.

and investment – affected? Barro says "no", given that the bequest motive is operative. Each dynasty will then choose the same consumption path $(c_{2t}, c_{1t})_{t=0}^{\infty}$ as it planned before the shift in fiscal policy. The reason is that the change in the time profile of lump-sum taxes will not make the dynasty feel richer or poorer. Aggregate consumption and saving in the economy will therefore remain unchanged.

The proof goes as follows. Suppose there are N dynasties in the economy, all alike. Since L_{t-1} is the total number of old agents in the economy in the current period, period t, each dynasty has L_{t-1}/N old members. Each dynasty must satisfy its intertemporal budget constraint. That is, the present value of its consumption stream cannot exceed the total wealth of the dynasty. In the optimal plan the present value of the consumption stream will equal the total wealth. Thus

$$\frac{L_{t-1}}{N}\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i}+(1+n)c_{1t+i}\right]=a_t+h_t, \tag{7.24}$$

where a_t is initial financial wealth of the dynasty and h_t is its human wealth (after taxes).[5] Multiplying through in (7.24) by N, we get the intertemporal budget constraint of the *representative dynasty*:

$$L_{t-1}\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i}+(1+n)c_{1t+i}\right]=Na_t+Nh_t\equiv A_t+H_t,$$

where A_t is aggregate financial wealth in the private sector and H_t is aggregate human wealth (after taxes):

$$H_t\equiv Nh_t=L_t\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i}-\tau_{t+i}). \tag{7.25}$$

The financial wealth consists of capital, K_t, and government bonds, B_t. Thus,

$$A_t+H_t=K_t+B_t+H_t. \tag{7.26}$$

Since B_t and H_t are the only terms in (7.26) involving taxes, we consider their sum:

$$\begin{aligned}B_t+H_t &= L_t\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(\tau_{t+i}-\frac{\gamma}{1+n}+w_{t+i}-\tau_{t+i})\\ &= L_t\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i}-\frac{\gamma}{1+n}),\end{aligned}$$

[5] How to get from the generation budget constraints, $c_{2t}+(1+n)b_t=(1+r)s_{t-1}$ and $(1+n)c_{1t}+(1+n)s_t=(1+n)(w_t-\tau_t+b_t)$, to the intertemporal budget constraint of the dynasty is shown in Appendix B.

where we have used (7.23) and (7.25). We see that the time profile of τ has disappeared and cannot affect $B_t + H_t$. Hence, the total wealth of the dynasty is unaffected by a change in the time profile of taxation.

As a by-product of the analysis we see that higher initial government debt has no effect on the sum, $B_t + H_t$, because H_t becomes equally much lower. This is what Barro (1974) meant by answering "no" to the question: "Are government bonds net [private] wealth?" (the title of his paper).

When the bequest motive is operative, Ricardian equivalence will also manifest itself in relation to a pay-as-you-go pension system. Suppose the mandatory contribution of the young (the workers) is raised in period t, before the old generation has decided the size of the bequest to be left to the young. Then, if the bequest motive is operative, the old generation will use the increase in their pension to leave higher bequests. In this way the young generation is compensated for the higher contribution they have to pay to the pay-as-you-go system. As a consequence all agents' consumption remain unchanged and so does resource allocation. Indeed, within this framework with perfect markets, as long as the bequest motive is operative, a broad class of lump-sum government fiscal actions can be nullified by offsetting changes in private saving and bequests.

Discussion

According to many macroeconomists, the modified golden rule and the Ricardian Equivalence result have elegance, but are hardly good approximations to reality:

1. The picture of the household sector as a set of dynasties, all alike, seems remote from reality. Universally, only a fraction of a country's population leave bequests.[6] In the last section of Chapter 6 we considered the "pure" case assumed in the Diamond OLG model where a bequest motive is entirely absent. In that case, because the new generations are then *new* agents, and the future taxes are levied partly on these new agents, the future taxes are no longer equivalent to current taxes. This is the *composition-of-the-tax-base argument* for Ricardian Non-equivalence. This argument is also relevant for "mixed" cases where only a fraction of the population leave bequests or where the bequests are motivated in other ways than assumed in the Barro model. Moreover, in a world of uncertainty bequests may simply be accidental rather than planned.

Even if there is a bequest motive of the altruistic form assumed by Barro, it will only be operative if it is strong enough, as we saw in Section 7.2.4.

2. When the bequest motive is not operative, Ricardian equivalence breaks down. Consider a situation where the constraint $b_{t+1} \geq 0$ is binding. Then

[6] Wolf (2002) found that in 1998 around 30 per cent of US households of "age" above 55 years (according to the age of the head of the household) reported to have received wealth transfer.

there will be no bequests. Parents would in fact like to pass on *debt* to the next generation. They are hindered by law, however. But the government can do what the private agents would like to do but cannot. Specifically, if for example $r_D < n$, there will be no bequests in the Barro economy and we have $r^* = r_D < n$ in steady state. So there is dynamic inefficiency. Yet the government can avoid this outcome and instead achieve $r^* = n$ within sight by, for example, a fiscal policy continually paying transfers to the old generation, financed by creation of public debt. The private agents cannot nullify this beneficial fiscal policy − and have no incentive to try doing so. However, the logical validity of this point notwithstanding, its practical relevance is limited since empirical evidence of dynamic inefficiency seems absent.[7]

3. Another limitation of Barro's analysis has to do with the dynastic-family story which portrays families as clearly demarcated and harmonious infinitely-lived entities. This view abstracts from the fact that:

(a) families are not formed by inbreeding, but by marriage of two individuals coming from two different parent families;

(b) the preferences of distinct family members may conflict; think for example of the schismatic family feud of the Chicago-based Pritzker family.[8]

Point (a) gives rise to a complex network of interconnected families. In principle, and perhaps surprisingly, this observation need not invalidate Ricardian equivalence. As Bernheim and Bagwell (1988) ironically remark, the problem is that "therein lies the difficulty". Almost all elements of fiscal policy, even on-the-face-of-it distortionary taxes, tend to become neutral. This is because virtually all the population is interconnected through chains involving parent-child linkages. Bernheim and Bagwell (1988, p. 311) conclude that in this setting, "Ricardian equivalence is merely one manifestation of a much more powerful and implausible neutrality theorem".[9]

Point (b) gives rise to broken linkages among the many linkages. This makes it difficult to imagine that Ricardian equivalence comes up.

4. From an econometric point of view, by and large Barro's hypothesis does not seem to do a good job in explaining how families actually behave. If altruistically linked extended family members really did pool their incomes across generations when deciding how much each should consume, then the amount that any particular family member consumes would depend only on the present discounted value of total future income stream of the extended family, not on that

[7] On the other hand, this latter fact *could* be a *consequence* of the described fiscal policy.

[8] The Pritzker family is one of the wealthiest American families, owning among other things the Hyatt hotel chain. In the early 2000s a long series of internal battles and lawsuits across generations resulted in the family fortune being split between 11 family members. For an account of this and other family owned business wars, see for example Gordon (2008).

[9] Barro (1989) answers the criticism of Bernheim and Bagwell.

person's share of that total. To state it differently: if the dynasty hypothesis were a good approximation to reality, then the ratio of the young's consumption to that of the parents should not depend systematically on the ratio of the young's *income* to that of the parents. But the empirical evidence goes in the opposite direction − own resources *do* matter (see Altonji et al. 1992 and 1997).

To conclude: The debt neutrality view is of interest as a theoretical benchmark case. In practice, however, tax cuts and debt financing by the government seem to make the currently alive generations feel wealthier and stimulate their consumption. Bernheim (1987) reviews the theoretical controversy and the empirical evidence of Ricardian equivalence. He concludes with the empirical estimate, for the US, that private saving offset only roughly half the decline in government saving that results from a shift from taxes to deficit finance.

We shall come across the issue of Ricardian equivalence or non-equivalence in other contexts later in this book.

7.4 The modified golden rule when there is technological progress*

Heretofore we have abstracted from technological progress. What does the modified golden rule look like when it is recognized that actual economic development is generally accompanied by technological progress?

To find out we extend the Barro model with Harrod-neutral technological progress. As we want consistency with Kaldor's stylized facts, we assume that technological progress is Harrod-neutral:

$$Y_t = F(K_t, \mathcal{T}_t L_t),$$

where F is a neoclassical aggregate production function with CRS and $\mathcal{T}_t$ (not to be confused with tax revenue T_t) is the technology level, which is assumed to grow exogenously at the constant rate $g > 0$, that is, $\mathcal{T}_t = \mathcal{T}_0(1+g)^t$, $\mathcal{T}_0 > 0$. Apart from this (and the specification of $u(c)$ below), everything is as in the simple Barro model analyzed above. Owing to equilibrium in the factor markets, K_t and L_t can be interpreted as predetermined variables, given from the supply side.

We have

$$\tilde{y}_t \equiv \frac{Y_t}{\mathcal{T}_t L_t} \equiv \frac{y_t}{\mathcal{T}_t} = F(\frac{K_t}{\mathcal{T}_t L_t}, 1) = F(\tilde{k}_t, 1) \equiv f(\tilde{k}_t), \qquad f' > 0, f'' < 0,$$

where $\tilde{k}_t \equiv K_t/(\mathcal{T}_t L_t) \equiv k_t/\mathcal{T}_t$. The dynamic aggregate resource constraint,

$K_{t+1} = (1-\delta)K_t + Y_t - C_t$, can now be written

$$\frac{K_{t+1}}{\mathcal{T}_t L_t} = (1+g)(1+n)\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + f(\tilde{k}_t) - \tilde{c}_t, \tag{7.27}$$

where $\tilde{c}_t \equiv C_t/(\mathcal{T}_t L_t) \equiv c_t/\mathcal{T}_t$, the per capita "technology-corrected" consumption level. With perfect competition we have the standard equilibrium relations

$$r_t = \frac{\partial Y_t}{\partial K_t} - \delta = \frac{\partial \left[\mathcal{T}_t L_t f(\tilde{k}_t)\right]}{\partial K_t} - \delta = f'(\tilde{k}_t) - \delta, \tag{7.28}$$

$$w_t = \frac{\partial Y_t}{\partial L_t} = \frac{\partial \left[\mathcal{T}_t L_t f(\tilde{k}_t)\right]}{\partial L_t} = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right]\mathcal{T}_t \equiv \tilde{w}(\tilde{k}_t)\mathcal{T}_t. \tag{7.29}$$

We want the model to comply with Kaldor's stylized facts. Thus, among other things the model should be consistent with a constant rate of return in the long run. Such a state requires $\tilde{k}_t$ to be constant, say equal to $\tilde{k}^*$; then $r_t = f'(\tilde{k}^*) - \delta \equiv r^*$. When $\tilde{k}_t$ is constant, then also $\tilde{c}_t$ and $\tilde{w}_t$ are constant, by (7.27) and (7.29), respectively. In effect, the capital-labor ratio k_t, output-labor ratio y_t, consumption-labor ratio c_t, and real wage w_t, all grow at the same rate as technology, the constant rate g. So a constant $\tilde{k}_t$ implies a balanced growth path with constant rate of return.

To be capable of maintaining a balanced growth path (and thereby be consistent with Kaldor' stylized facts) when the bequest motive is operative, the Barro model needs that the period utility function, $u(c)$, is a CRRA function. Indeed, when the bequest motive is operative, the Barro model becomes essentially a representative agent model in spite of its OLG structure. And it can be shown (see Appendix C) that for existence of a balanced growth path in a representative agent model with Harrod-neutral technological progress, we have to assume that the period utility function, $u(c)$, is of CRRA form:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0, \tag{7.30}$$

where θ is the constant (absolute) elasticity of marginal utility (if $\theta = 1$, (7.30) should be interpreted as $u(c) = \ln c$).

From now on we shall often write the CRRA utility function this way, i.e., without adding the "normalizing" constant $-1/(1-\theta)$. This is immaterial for the resulting consumption/saving behavior, but we save notation and avoid the inconvenience that an infinite sum of utilities, as in (7.3), may not be bounded for the sole reason that an economically trivial constant has been added to the crucial part of the period utility function.

When the bequest motive is operative, the young parent's optimality condition (7.10) is also valid in the new situation with technological progress. Assuming (7.30), we can write (7.10) as $c_{1t}^{-\theta} = c_{1t+1}^{-\theta}(1 + r_{t+1})/(1 + R)$, where we have used that $(1 + \bar{R})(1 + n) \equiv 1 + R$; this implies

$$(\frac{c_{1t+1}}{c_{1t}})^{\theta} = \frac{1 + r_{t+1}}{1 + R}, \tag{7.31}$$

where c_{1t+1}/c_{1t} is 1+ the growth rate (as between generations) of consumption as young. To the extent that the right-hand side of (7.31) is above one, it expresses the excess of the rate of return over and above the intergenerational discount rate. The interpretation of (7.31) is that this excess is needed for $c_{1t+1}/c_{1t} > 1$. For the young to be willing to save and in the next period leave positive bequests, the return on saving must be large enough to compensate the parent for the absence of consumption smoothing (across time as well as across generations). The larger is θ (the desire for consumption smoothing), for a given $c_{1t+1}/c_{1t} > 1$, the larger must r_{t+1} be in order to leave the parent satisfied with not consuming more herself. In the same way, the larger is c_{1t+1}/c_{1t} (and therefore the inequality across generations), for a given θ, the larger must r_{t+1} be in order to leave the parent satisfied with not consuming more.

As observed in connection with (7.14), the old at the beginning of period $t+1$ own all capital in the economy. So the aggregate capital stock equals their saving in the previous period: $K_{t+1} = s_t L_t$. Defining $\tilde{s}_t \equiv s_t/\mathcal{T}_t$, we get

$$\tilde{k}_{t+1} = \frac{\tilde{s}_t}{(1 + g)(1 + n)}. \tag{7.32}$$

Steady state

By (7.32), in steady state $\tilde{s}_t = (1+g)(1+n)\tilde{k}^* \equiv \tilde{s}^*$. The consumption per young and consumption per old in period t add to total consumption in period t, that is, $C_t = L_t c_{1t} + L_t(1 + n)^{-1}c_{2t}$. Dividing through by effective labor gives

$$\tilde{c}_t \equiv \frac{C_t}{\mathcal{T}_t L_t} = \tilde{c}_{1t} + (1 + n)^{-1}\tilde{c}_{2t},$$

where $\tilde{c}_{1t} \equiv c_{1t}/\mathcal{T}_t$ and $\tilde{c}_{2t} \equiv c_{2t}/\mathcal{T}_t$. In this setting we define a steady state as a path along which not only $\tilde{k}_t$ and $\tilde{c}_t$, but also $\tilde{c}_{1t}$ and $\tilde{c}_{2t}$ separately, are constant, say equal to $\tilde{c}_1^*$ and $\tilde{c}_2^*$, respectively. Dividing through by $\mathcal{T}_t$ in the two period budget constraints, (7.4) and (7.5), we get $\tilde{b}_t \equiv b_t/\mathcal{T}_t = \tilde{c}_1^* + \tilde{s}^* - \tilde{w}(\tilde{k}^*) \equiv \tilde{b}^*$. Consequently, in a steady state with an operative bequest motive, bequest per young, b_t, consumption per young, c_{1t}, saving per young, s_t, and consumption per old, c_{2t}, all grow at the same rate as technology, the constant rate g.

But how are $\tilde{k}^*$ and r^* determined? Inserting the steady state conditions $c_{1t+1} = c_{1t}(1+g)$ and $r_{t+1} = r^*$ into (7.31) gives

$$(1+g)^\theta = \frac{1+r^*}{1+R}$$

or

$$1 + r^* = (1+R)(1+g)^\theta \equiv (1+\bar{R})(1+n)(1+g)^\theta. \tag{7.33}$$

This is the *modified golden rule* when there is Harrod-neutral technological progress at the rate g and a constant (absolute) elasticity of marginal utility of consumption θ. The modified golden rule says that for the economy to be in a steady state with positive bequests, it is necessary that the gross interest rate matches the "subjective" gross discount rate, taking account of both (a) the own-generation preference rate R, and (b) the fact that there is aversion (measured by θ) to the lack of consumption smoothing arising from per capita consumption growth at rate g. If in (7.33) "=" were replaced by ">" ("<"), then *more* (less) saving and capital accumulation would take place, tending to push the system away from a steady state. When $g = 0$ (no technological progress), (7.33) reduces to the simple modified golden rule, (7.15).

The effective capital-labor ratio in steady state, $\tilde{k}^*$, satisfies $f'(\tilde{k}^*) - \delta = r^*$, where r^* is given from the modified golden rule, (7.33), when the bequest motive is operative. Assuming f satisfies the Inada conditions, this equation has a (unique) solution $\tilde{k}^* = f'^{-1}(r^* + \delta) = f'^{-1}((1+R)(1+g)^\theta - 1 + \delta) \equiv \tilde{k}_{MGR}$. Since $f'' < 0$, it follows that the higher are R and g, the lower is the modified-golden-rule capital intensity, $\tilde{k}_{MGR}$.

To ensure that the infinite sum of discounted lifetime utilities is bounded from above along the steady state path (so that maximization is possible) we need an *effective* intergenerational discount rate, $\bar{R} \equiv (1+R)/(1+n) - 1$, that is not only positive, but sufficiently large. In the next chapter it is shown that $1 + \bar{R} > (1+g)^{1-\theta}$ is required and that this inequality also ensures that the transversality condition (7.8) holds in a steady state with positive bequests. The required inequality is equivalent to

$$1 + R > (1+n)(1+g)^{1-\theta}, \tag{7.34}$$

which we assume satisfied, in addition to $R > n$.[10] Combining this with (7.33), we thus have

$$1 + r^* = (1+R)(1+g)^\theta > (1+n)(1+g). \tag{7.35}$$

[10] Fortunately, (7.34) is more strict than the restriction $R > n$, used up to now, only if $\theta < 1$, which is not the empirically plausible case.

Let $\tilde{k}_{GR}$ denote the *golden rule* capital intensity, defined by $1 + f'(\tilde{k}_{GR}) - \delta = (1 + n)(1 + g)$.[11] Since $f'' < 0$, we conclude that $\tilde{k}_{MGR} < \tilde{k}_{GR}$. In the "unmodified" golden rule with technological progress, the interest rate (reflecting the net marginal productivity of capital) equals the output growth rate, which with technological progress is $(1+n)(1+g)-1$. The "modification" displayed by (7.33) comes about because both the strictly positive effective intergenerational discount rate $\bar{R}$ and the elasticity of marginal utility enter the determination of r^*. In view of the parameter inequality (7.34), the intergenerational discounting results in a lower effective capital-labor ratio and higher rate of return than in the golden rule. Indeed, in general equilibrium with positive bequests it is impossible for the economy to reach the golden rule.

The condition for positive bequests

Using that $u'(c_{2t}) = c_{2t}^{-\theta} = (\tilde{c}_2^* \mathcal{T}_t)^{-\theta}$ in steady state, the proof of Proposition 1 can be extended to show that bequests are positive in steady state if and only if

$$1 + R = (1 + r^*)(1 + g)^{-\theta} < (1 + r_D)(1 + g)^{-\theta}, \tag{7.36}$$

where the equality follows from (7.35), and r_D is the steady-state interest rate in the associated "well-behaved" Diamond economy. It can moreover be shown that if both (7.36) and (7.34) (as well as $R > n$) hold, and the initial $\tilde{k}$ is in a neighborhood of the modified-golden-rule value, then the bequest motive is operative in every period and the economy converges over time to the modified-golden-rule steady state. This stability result is shown in the next chapter.

Intuition might make us think that a higher rate of technological progress, g, would make the old more reluctant to leave bequests since they know that the future generations will benefit from better future technology. For fixed r_D, this intuition is confirmed by (7.36). The inequality shows that for fixed r_D, a higher rate of technological progress implies a lower cut-off value for R. But r_D is not fixed but an increasing function of g. That is, whether or not it holds that a higher g implies a lower cut-off value for R, depends on which effect is the stronger one, the direct effect in (7.36) of the higher g or the indirect effect through the rise in r_D. See Exercise 7.??

Calibration We shall give a rough informal estimate of the intergenerational discount rate, R, by the method of calibration. Generally, *calibration* means to choose parameter values such that the model matches a list of data characteristics.[12] Given the formula (7.33), we want our estimate of R to be consistent with

[11] See Chapter 4.

[12] A next step (not pursued here) is to consider other data and check whether the model also fits them.

a long-run annual real rate of return, $\tilde{r}^*$, of about 0.05, when reasonable values for the annual rate of technological progress, $\tilde{g}$, and the elasticity of marginal utility of consumption, θ, are chosen. For advanced economies after the Second World War, values with some empirical support are $\tilde{g} = 0.02$ and $\theta \in (1, 5)$. Choosing $\theta = 2$ and taking into account that our model has an implicit period length of about 30 years, we get:

$$\begin{aligned} 1 + g &= (1 + \tilde{g})^{30} = 1.02^{30} = 1.8114 \\ 1 + r^* &= (1 + \tilde{r}^*)^{30} = 1.05^{30} = 4.3219 \\ 1 + R &= \frac{1 + r^*}{(1 + g)^{\theta}} = \frac{4.3219}{1.8114^2} = 1.3172 \end{aligned}$$

Thus $R = 0.3172$. On a yearly basis the corresponding intergenerational discount rate is then $\tilde{R} = (1 + R)^{1/30} - 1 \simeq 0.0092$. As to $\bar{R}$, with $\tilde{n} = 0.008$, we get $1 + n = (1 + \tilde{n})^{30} = 1.008^{30} = 1.2700$, so that $1 + \bar{R} \equiv (1 + R)/(1 + n) = 1.0371$. This gives an *effective* intergenerational discount rate on an *annual* basis equal to 0.0012.[13]

Do these numbers indicate that we are in a situation where the bequest motive is operative? The answer would be affirmative if and only if $r^* < r_D$, cf. (7.36). Whether the latter inequality holds, depends on the time preference rate, ρ, and the aggregate production function, f. Our empirical knowledge about both is limited. Exercise 7.? considers the Cobb-Douglas case for alternative values of $\theta = 1$.

7.5 Concluding remarks

We have studied Barro's model of overlapping generations linked through altruistic bequests. Barro's insight is that intergenerational altruism may extend households' planning horizon. If the extent of altruism is sufficiently high so that the bequest motive (in the Barro form) is operative in every period, then the model

- becomes a representative agent model and implies Ricardian equivalence;
- results in a simple formula for the long-run real interest rate (the modified golden rule).

What is the implication for intergenerational distribution of welfare? Even ignoring technological progress, a strictly positive intergenerational discount rate,

[13] Control of the calculation: $\frac{1+\tilde{r}^*}{(1+\tilde{n})(1+\tilde{g})^2} = \frac{1.05}{1.008 \cdot 1.02^2} = 1.0012$, hence OK.

R, does not imply that future generations must end up worse off than current generations. This is because positivity of R does not hinder existence of a stable steady state. The role of R is to determine *what* steady state the economy is heading to, that is, what effective capital-labor ratio and level of per capita consumption is sustainable. Within the constraint displayed by (7.34), a higher R results in a lower steady-state capital intensity, a lower level of per capita consumption (in an economy without technological progress), and a lower position of the upward-sloping time path of per capita consumption in an economy with Harrod-neutral technological progress.

The simplifying assumption behind the modified-golden-rule formula for the long-run interest rate, the representative agent assumption, has been seriously questioned. The modified-golden-rule formula itself finds no empirical support. From the formula (7.35) we should expect to find positive comovements over the medium run between the rate of interest (rate of return) and the productivity growth rate. However, the investigation by Hamilton et al. (2016), covering more than a century and many countries, finds no relationship of that kind.

In the Barro model resource allocation and the coordination of economic behavior across generations is brought about through the market mechanism and an operative bequest motive due to parental altruism. In the next chapter we shall study a situation where the coordination across generations is brought about by a fictional social planner maximizing a social welfare function.

7.6 Literature notes

(incomplete)

The criticism by Bernheim and Bagwell is answered in Barro (1989). A survey of the debt neutrality issues is provided by Dalen (1992), emphasizing "demographic realism".

We have concentrated on the Barro model where bequests reflect the concern of parents for the welfare of their offspring. The Barro model was further developed and analyzed by Buiter (1980), Abel (1987), and Weil (1987) and our treatment above draw on these contributions. Our analysis ruled out circumstances where children help support their parents. This is dealt with in Abel (1987) and Kimball (1987); see also the survey in Bernheim (1987). For a more general treatment of bequests in the economy, see for example Laitner (1997).

Reviews of how to model the distinction between "life-cycle wealth" and "inherited wealth" and of diverging views on the empirical importance of inherited wealth are contained in Kessler and Masson (1988) and Malinvaud (1998a). How much of net wealth accumulation in Scandinavia represents intended transfers and bequests is studied by Laitner and Ohlsson (2001) and Danish Economic

Council (2004).

7.7 Appendix

A. The envelope theorem for an unconstrained maximum

In the solution of the parent's decision problem in the Barro model of Section 7.2 we appealed to the envelope theorem which, in its simplest form, is the principle that in an interior maximum the total derivative of a maximized function w.r.t. a parameter equals the partial derivative w.r.t. that parameter. More precisely:

ENVELOPE THEOREM Let $y = f(a, x)$ be a continuously differentiable function of two variables. The first variable, a, is conceived as a parameter and the other variable, x, as a control variable. Let $g(a)$ be a value of x at which $\frac{\partial f}{\partial x}(a, x) = 0$, i.e., $\frac{\partial f}{\partial x}(a, g(a)) = 0$. Let $F(a) \equiv f(a, g(a))$. Provided $F(a)$ is differentiable, we have

$$F'(a) = \frac{\partial f}{\partial a}(a, g(a)),$$

where $\partial f/\partial a$ denotes the partial derivative of $f(\cdot)$ w.r.t. the first argument.

Proof $F'(a) = \frac{\partial f}{\partial a}(a, g(a)) + \frac{\partial f}{\partial x}(a, g(a))g'(a) = \frac{\partial f}{\partial a}(a, g(a))$, since $\frac{\partial f}{\partial x}(a, g(a)) = 0$ by definition of $g(a)$. □

That is, when calculating the total derivative of a function w.r.t. a parameter and evaluating this derivative at an interior maximum w.r.t. a control variable, the envelope theorem allows us to ignore the second term arising from the chain rule. This is also the case if we calculate the total derivative at an interior minimum. Extension to a function of n control variables is straightforward.[14]

The envelope theorem in action For convenience we repeat the first-order conditions (7.6) and (7.7):

$$u'(c_{1t}) = (1+\rho)^{-1}u'(c_{2t+1})(1 + r_{t+1}), \quad (*)$$
$$(1+\rho)^{-1}u'(c_{2t+1}) = (1+\bar{R})^{-1}u'(c_{1t+1})(1+n)^{-1}. \quad (**)$$

We described in Section 7.2 how the parent chooses s_t and b_{t+1} so as to maximize the objective function $\tilde{U}_t$, taking into account the descendants' optimal responses to the received bequest b_{t+1}. We claimed without proof that in view of the envelope theorem, these two at first sight incomplete first-order conditions

[14] For extensions and more rigorous framing of the envelope theorem, see for example Sydsaeter et al. (2006).

are correct, both as they read and for t replaced by $t+i$, $i = 1, 2, ...$, and do indeed characterize an optimal plan.

To clarify the issue we substitute the two period budget constraints of a young into the objective function to get

$$\begin{aligned} \tilde{U}_t(s_t, b_{t+1}) &= u(w_t + b_t - s_t) + (1+\rho)^{-1}u((1+r)s_t \\ &\quad -(1+n)b_{t+1}) + (1+\bar{R})^{-1}\tilde{U}_{t+1}(\hat{s}_{t+1}, \hat{b}_{t+2}), \end{aligned} \tag{7.37}$$

where $\hat{s}_{t+1}$ and $\hat{b}_{t+2}$ are the optimal responses of the next generation and where $\tilde{U}_{t+1}(\cdot)$ can be written in the analogue recursive way, and so on for all future generations. The responses of generation $t+1$ are functions of the received b_{t+1} so that we can write

$$\hat{s}_{t+1} = \hat{s}(b_{t+1}, t+2), \qquad \hat{b}_{t+2} = \hat{b}(b_{t+1}, t+2),$$

where the second argument, $t+2$, represents the influence of w_{t+1} and r_{t+2}.

Our at first sight questionable approach rests on the idea that the smooth function $\tilde{U}_{t+1}(\cdot)$ is flat at an interior maximum so that any small change in the descendants' optimal responses induced by a small change in b_{t+1} has a negligible effect on the value of the function, hence also on the value of $\tilde{U}_t(\cdot)$. A detailed argument goes as follows.

For the first-order conditions (*) and (**), both as they read and for t replaced by $t+i$, $i = 1, 2, ...$, to make up a correct characterization of optimal behavior by a parent who takes the optimal responses by the descendants into account, the first-order conditions must imply that the *total* derivative of the parent's objective function w.r.t. b_{t+1} vanishes. To see whether our "half-way" optimization procedure has ensured this, we first forward the period budget constraint (7.5) one period to get:

$$c_{2t+2} + (1+n)\hat{b}_{t+2} = (1+r_{t+2})\hat{s}_{t+1}. \tag{7.38}$$

Using this expression we substitute for c_{2t+2} in (7.37) and let the function $\hat{U}_t(s_t, b_{t+1}, \hat{s}_{t+1}, \hat{b}_{t+2})$ represent the right-hand side of (7.37).

Although the parent chooses both s_t and b_{t+1}, only the choice of b_{t+1} affects the next generation. Calculating the total derivative of $\hat{U}_t(\cdot)$ w.r.t. b_{t+1}, we get

$$
\begin{aligned}
&d\hat{U}(s_t, b_{t+1}, \hat{s}_{t+1}, \hat{b}_{t+2})/db_{t+1} = \\
&\quad (1+\rho)^{-1}u'(c_{2t+1})\left(-(1+n)\right) + (1+\bar{R})^{-1}u'(c_{1t+1})(1 - \frac{\partial \hat{s}_{t+1}}{\partial b_{t+1}}) \\
&\quad +(1+\bar{R})^{-1}\left\{(1+\rho)^{-1}u'(c_{2t+2})\left((1+r_{t+2})\frac{\partial \hat{s}_{t+1}}{\partial b_{t+1}} - (1+n)\frac{\partial \hat{b}_{t+2}}{\partial b_{t+1}}\right)\right. \\
&\quad \left. +(1+\bar{R})^{-1}u'(c_{1t+2})\frac{\partial \hat{b}_{t+2}}{\partial b_{t+1}}\right\} + ... \\
&= (1+\rho)^{-1}u'(c_{2t+1})\left(-(1+n)\right) + (1+\bar{R})^{-1}u'(c_{1t+1}) \\
&\quad -(1+\bar{R})^{-1}\left[u'(c_{1t+1}) - (1+\rho)^{-1}u'(c_{2t+2})(1+r_{t+2})\right]\frac{\partial \hat{s}_{t+1}}{\partial b_{t+1}} \\
&\quad +(1+\bar{R})^{-1}\left[(1+\rho)^{-1}u'(c_{2t+2})\left(-(1+n)\right) + (1+\bar{R})^{-1}u'(c_{1t+2})\right]\frac{\partial \hat{b}_{t+2}}{\partial b_{t+1}} + ... \\
&= (1+\rho)^{-1}u'(c_{2t+1})\left(-(1+n)\right) + (1+\bar{R})^{-1}u'(c_{1t+1}) = 0. \qquad (7.39)
\end{aligned}
$$

The second last equality sign is due to the first-order conditions (*) and (**), first with t replaced by $t+1$, implying that the two terms in square brackets vanish, second with t replaced by $t+i$, $i = 2, 3, \ldots$, implying that also all the remaining terms, represented by "...", vanish (since the latter terms can be written in the same way as the former). The last equality sign is due to (**) as it reads. Thus, also the *total* derivative is vanishing as it should at an interior optimum.

Note that the expression in the last line of the derivation is the *partial* derivative of $\hat{U}_t(\cdot)$, namely $\partial\hat{U}(s_t, b_{t+1}, \hat{s}_{t+1}, \hat{b}_{t+2})/\partial b_{t+1}$. The whole derivation is thus a manifestation of the envelope theorem for an unconstrained maximum: in an interior optimum the total derivative of a maximized function w.r.t. a parameter, here b_{t+1}, equals the partial derivative w.r.t. that parameter.

B. The intertemporal budget constraint of a dynasty

We here show how to derive a dynasty's intertemporal budget constraint as presented in (7.24) of Section 7.3. With lump-sum taxation and constant interest rate, r, the period budget constraints of a member of generation t are

$$
\begin{aligned}
c_{1t} + s_t &= w_t - \tau_t + b_t, \qquad \text{and} \qquad (7.40) \\
c_{2t+1} + (1+n)b_{t+1} &= (1+r)s_t. \qquad (7.41)
\end{aligned}
$$

We isolate s_t in (7.41), substitute into (7.40), and reorder to get

$$
b_t = c_{1t} + \frac{c_{2t+1}}{1+r} - (w_t - \tau_t) + \frac{1+n}{1+r}b_{t+1}.
$$

Then, by forward substitution,

$$\begin{aligned} b_t &= \sum_{i=0}^{j}(\frac{1+n}{1+r})^i\left[c_{1t+i}+\frac{c_{2t+i+1}}{1+r}-(w_{t+i}-\tau_{t+i})\right]+(\frac{1+n}{1+r})^{j+1}b_{t+j+1} \\ &= \sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i\left[c_{1t+i}+\frac{c_{2t+i+1}}{1+r}-(w_{t+i}-\tau_{t+i})\right], \qquad (7.42) \end{aligned}$$

assuming $\lim_{j\to\infty}(\frac{1+n}{1+r})^{j+1}b_{t+j+1}=0$, in view of $r>n$. For every old in any given period there are $1+n$ young. We therefore multiply through in (7.42) by $1+n$ and reorder:

$$(1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(c_{1t+i}+\frac{c_{2t+i+1}}{1+r})=(1+n)b_t+(1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(w_{t+i}-\tau_{t+i}).$$

To this we add the period budget constraint of an old member of the dynasty,

$$c_{2t}+(1+n)b_t=(1+r)s_{t-1},$$

and get the consolidated intertemporal budget constraint of the dynasty in period t:

$$c_{2t}+(1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(c_{1t+i}+\frac{c_{2t+i+1}}{1+r})=(1+r)s_{t-1}+(1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(w_{t+i}-\tau_{t+i}),$$

where $(1+n)b_t$ has been cancelled out on both sides. Dividing through by $1+r$ and reordering gives

$$\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i}+(1+n)c_{1t+i}\right]=s_{t-1}+(1+n)\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i}-\tau_{t+i}).$$

This is the intertemporal budget constraint, as seen from the beginning of period t, of a dynasty with one old member in period t. With L_{t-1} old members, this becomes

$$\begin{aligned} & L_{t-1}\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i}+(1+n)c_{1t+i}\right] \\ = \;& s_{t-1}L_{t-1}+L_t\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i}-\tau_{t+i})=A_t+H_t, \end{aligned}$$

where $A_t=s_{t-1}L_{t-1}$ is the financial wealth in the beginning of period t and H_t is the human wealth, as defined in (7.25). Dividing through by N gives (7.24).

C. Proof that a representative agent model allows balanced growth only if the period utility function is CRRA

This appendix refers to Section 7.4. When the bequest motive is operative, the Barro model becomes a representative agent model where the intergenerational Euler equation (7.10) holds for all dynasties and therefore also at the aggregate level. For convenience we repeat the Euler equation in question here:

$$u'(c_{1t}) = \frac{1 + r_{t+1}}{1 + R} u'(c_{1t+1}), \tag{7.43}$$

in view of $(1 + \bar{R})^{-1} \equiv (1 + R)^{-1}(1 + n)$. In balanced growth, $c_{1t+1} = (1 + g)c_{1t}$, where $g > 0$ and $r_{t+1} = r^*$, so that (7.43) takes the form

$$u'(c_{1t}) = \frac{1 + r^*}{1 + R} u'((1 + g)c_{1t}) \equiv \omega(c_{1t}), \tag{7.44}$$

which must hold for all $c_{1t} > 0$ to be generally consistent with balanced growth. Thus, the derivatives on both sides should also be equal for all $c_{1t} > 0$:

$$u''(c_{1t}) = \omega'(c_{1t}) = \frac{1 + r^*}{1 + R} u''((1 + g)c_{1t})(1 + g). \tag{7.45}$$

Dividing through by $u'(c_{1t})$ in accordance with (7.44) and multiplying by c_{1t} yields

$$\frac{c_{1t}u''(c_{1t})}{u'(c_{1t})} = \frac{(1 + g)c_{1t}u''((1 + g)c_{1t})}{u'((1 + g)c_{1t})},$$

showing that for all $c_{1t} > 0$, the (absolute) elasticity of marginal utility should be the same at the consumption level c_{1t} as at the consumption level $(1 + g)c_{1t}$. It follows that $u(\cdot)$ must be such that the (absolute) elasticity of marginal utility, $\theta(c) \equiv cu''(c)/u'(c)$, is independent of c, i.e., $\theta(c) = \theta > 0$. We know from Chapter 3 that this requires that $u(\cdot)$, up to a positive linear transformation, has the CRRA form $c^{1-\theta}/(1 - \theta)$.

7.8 Exercises