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A suggested solution to the problem set
at the exam in
Advanced Macroeconomics
January 21, 2013

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's perform-ance demonstrates (a) accurate and thorough understanding of the concepts, methods, and models in the course, (b) knowledge of the major empirical regularities for aggreg-ate economic variables, and (c) ability to use these theoretical tools and this empirical knowledge to answer macroeconomic questions.

1. Solution to Problem 1

The given equations from the Blanchard OLG model are:

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \frac{\lambda+b}{b}\tilde{c}_t - (\delta+g+b-m)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \qquad (1.1)$$

$$\dot{\tilde{c}}_t = \left[f'(\tilde{k}_t) - \delta - \rho + \lambda - g\right]\tilde{c}_t - b(\rho+m)\tilde{k}_t, \qquad (1.2)$$

together with the condition that for any fixed pair (v, t_0), where $t_0 \geq 0$ and $v \leq t_0$,

$$\lim_{t\to\infty} a_{v,t}e^{-\int_{t_0}^{t}(f'(\tilde{k}(s))-\delta+m)ds} = 0. \qquad (1.3)$$

Notation: $\tilde{k}_t \equiv K_t/(T_tL_t)$ and $\tilde{c}_t \equiv C_t/(T_tN_t) \equiv c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively; N_t is population, L_t is labor supply, and T_t is the technology level, all at time t; f is a production function on intensive form, satisfying $f(0) = 0$, $f' > 0$, $f'' < 0$, and the Inada conditions. Finally, $a_{v,t}$ is financial wealth at time t of an individual born at time v. The remaining symbols stand for parameters and we assume all these are strictly positive. Furthermore, $\rho \geq b - m \geq 0$ and $\lambda < \delta + \rho + g$.

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a)   Parameters: λ = retirement rate, b = birth rate, δ = capital depreciation rate, g = growth rate of technology, m = mortality rate, ρ = pure rate of time preference (utility discount rate, a measure of impatience). The model relies on the simplifying assumption that for a given individual the probability of having a remaining lifetime, X, longer than some arbitrary number x is $P(X > x) = e^{-mx}$, the same for all (i.e., independent of age). Individual labor supply is assumed to decline exponentially at rate λ with increasing age. Both population and labor force grow at the rate $n \equiv b - m$.

The equation (1.1) is essentially just national income accounting for a closed economy. Isolating $f(\tilde{k}_t)$ on one side we have aggregate gross income per unit of effective labor on this side and consumption plus gross investment per unit of effective labor on the other side. Because the technology-corrected capital-labor ratio, $\tilde{k}_t$, has employment, L_t, in the denominator while the technology-corrected per-capita consumption, $\tilde{c}_t$, has population, N_t, in the denominator, $\tilde{c}_t$ is multiplied by the inverse of the participation rate, $L_t/N_t = b/(\lambda + b)$.

As to the first term on the right-hand side of (1.2), notice that instantaneous utility in the Blanchard OLG model is logarithmic, so that the individual Keynes-Ramsey rule at time t for a person born at time v is simply

$$\frac{\partial c_{v,t}}{\partial t} = [r_t + m - (\rho + m)]\, c_{v,t} = (r_t - \rho)c_{v,t}. \tag{1.4}$$

In general equilibrium with perfect competition $r_t = f'(\tilde{k}_t) - \delta$. The corresponding Keynes-Ramsey rule for growth-corrected per-capita consumption is therefore $\dot{\tilde{c}}_t = \left[f'(\tilde{k}_t) - \delta - \rho - g\right]\tilde{c}_t$. But due to the gradual replacement of dying elder individuals with low labor supply by younger individuals supplying more labor, the first term on the right-hand side of (1.2) also includes $+\lambda\tilde{c}_t$.

The second term on the right-hand side of (1.2) represents another aspect of the generation replacement. The arrival of newborns is Nb per time unit. The fact that they have *more human* wealth than those who they displace has already been taken into account by the mentioned $+\lambda\tilde{c}_t$. But the newborns enter the economy with *less financial* wealth than the "average citizen". This lowers aggregate consumption by $b(\rho + m)A_t$ per time unit, where A_t is aggregate private financial wealth. In general equilibrium in the closed economy (without government debt) we have $A_t = K_t$. Correcting for population and technology growth, we end up with a lowering of $\tilde{c}_t$ equal to $b(\rho + m)\tilde{k}_t$. This explains the second term in (1.2).

Finally, (1.3) is a transversality condition as seen from time t_0 for a person born at time v. The condition says that the No-Ponzi-Game condition is not over-satisfied (a
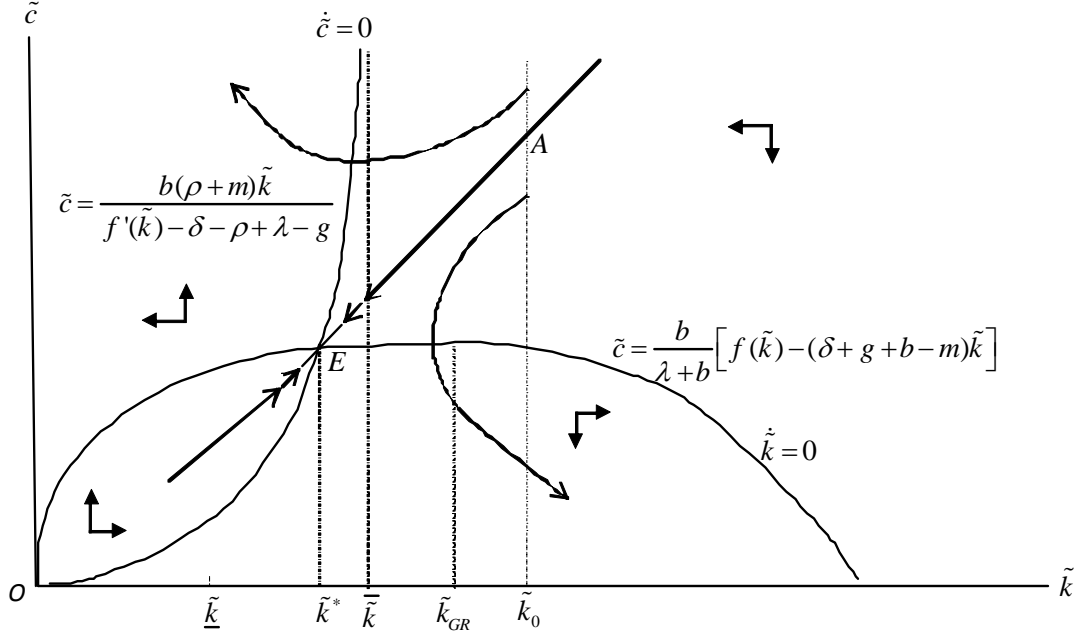
Figure 1.1:

necessary condition for individual optimality).

b)   The equation describing the $\dot{\tilde{k}} = 0$ locus is

$$\tilde{c} = \frac{b}{\lambda + b} \left[f(\tilde{k}) - (\delta + g + b - m)\tilde{k} \right]. \tag{1.5}$$

The equation describing the $\dot{\tilde{c}} = 0$ locus is

$$\tilde{c} = \frac{b(\rho + m)\tilde{k}}{f'(\tilde{k}) - \delta - \rho + \lambda - g}. \tag{1.6}$$

Let $\overline{\tilde{k}}$ be defined by

$$f'(\overline{\tilde{k}}) - \delta = \rho - \lambda + g. \tag{1.7}$$

That is, $\overline{\tilde{k}}$ is defined as the value of $\tilde{k}$ at which the denominator of (1.6) vanishes. Such a value exists since, in addition to the Inada conditions, the inequality

$$\lambda < \delta + \rho + g$$

is assumed to hold. Another key value of $\tilde{k}$ is the golden-rule value, $\tilde{k}_{GR}$, determined by the requirement

$$f'\left(\tilde{k}_{GR}\right) - \delta = n + g, \qquad \text{where } n = b - m.$$

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The phase diagram and the $\dot{\tilde{k}} = 0$ and $\dot{\tilde{c}} = 0$ loci are shown in Fig. 1.1. The $\dot{\tilde{c}} = 0$ locus is everywhere to the left of the line $\tilde{k} = \overline{\overline{k}}$ and is asymptotic to this line for $\tilde{k} \to \overline{\overline{k}}$. The figure also shows the steady-state point, E, where the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus. The corresponding capital intensity is $\tilde{k}^*$, to which corresponds the (growth-corrected) per-capita consumption level $\tilde{c}^*$. Fig. 1.1 depicts a case where $\overline{\overline{k}} \leq \tilde{k}_{GR}$ so that $\tilde{k}^* < \tilde{k}_{GR}$, that is, the economy is dynamically efficient. Yet, since we may have $b - m \leq \rho < \lambda + b - m$, so that $\overline{\overline{k}} > \tilde{k}_{GR}$, dynamic inefficiency cannot be ruled out theoretically (a typical feature of an OLG model).

The directions of movement in the different regions of the phase diagram are determined by the differential equations (1.1) and (1.2); the directions are shown by arrows. The arrows taken together show that the steady state, E, is a saddle point. Moreover, we have one predetermined variable, $\tilde{k}$, and one jump variable, $\tilde{c}$, the saddle path is not parallel to the jump-variable axis, and the diverging paths can be ruled out as equilibrium paths (see below). Hence the steady state is saddle-point stable.

The saddle path is the only path that satisfies *all* the conditions of general equilibrium (individual utility maximization for given expectations, profit maximization by firms, continuous market clearing, and perfect foresight). The other paths in the diagram are diverging and either violate the transversality conditions of the individuals (paths that in the long run point South-East) or the NPG conditions of the individuals[2] (paths that in the long run point North-West). Hence, equilibrium initial consumption is determined as the ordinate, $\tilde{c}_0$, to the point where the vertical line $\tilde{k} = \tilde{k}_0$ crosses the saddle path. Over time the economy moves from this point, along the saddle path, towards the steady state.

c)   In view of stability we have for $t \to \infty$,

$$r_t = f'(\tilde{k}_t) - \delta \to f'(\tilde{k}^*) - \delta \equiv r^*, \tag{1.8}$$

where r^* is the long-run rate of return. From the definition of $\overline{\overline{k}}$ and the fact that $\overline{\overline{k}} > \tilde{k}^*$, follows

$$f'(\overline{\overline{k}}) - \delta = \rho + g - \lambda < f'(\tilde{k}^*) - \delta = r^*,$$

in view of $f'' < 0$. This gives a lower end point of the interval to which r^* belongs.

d)   As to the upper end point of the interval, we are told that $\tilde{k}^* > \underline{\tilde{k}} > 0$, where $\underline{\tilde{k}}$

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[2] And therefore also the transversality conditions.

satisfies $f'(\tilde{\underline{k}}) - \delta = \rho + g + b$. It follows that

$$r^* = f'(\tilde{k}^*) - \delta < \rho + g + b,$$

in view of $f'' < 0$. This gives an upper end point of the interval to which r^* belongs. Hence,

$$\rho + g - \lambda < r^* < \rho + g + b. \tag{1.9}$$

e) In a Ramsey model with logarithmic instantaneous utility function (hence $\theta = 1$) and the same ρ and g as in the present model, the long-run rate of return is

$$r_R^* = \rho + \theta g = \rho + g.$$

In our Blanchard OLG model the long-run interest rate may differ from this value because of two effects of life-cycle behavior, that go in opposite directions. On the one hand, finite lifetime leads to a higher effective utility discount rate, hence *less* saving and therefore a tendency for $r^* > \rho + g$. On the other hand, retirement entails an incentive to save *more* (for the late period in life with low labor income). This results in a tendency for $r^* < \rho + g$, everything else equal. Without further information about parameters and technology, we can not say whether $r^* > r_R^*$ or $r^* < r_R^*$.

f) The growth rate of individual consumption through lifetime is given from the Keynes-Ramsey rule, (1.4), which in steady state yields

$$\frac{\dot{c}_{v,t}}{c_{v,t}} = r^* - \rho \equiv g_{c_{v,t}}^* > g - \lambda,$$

in view of (1.9). The growth rate of per capita consumption, $c_t \equiv \tilde{c}_t T_t$, is $\dot{c}_t/c_t = \dot{\tilde{c}}_t/\tilde{c}_t + g$, which in steady state gives

$$\frac{\dot{c}_t}{c_t} = g \equiv g_{c_t}^*.$$

Because $L_t/N_t = b/(\lambda + b)$, it is clear that per capita consumption will in steady state grow at the same rate as labor productivity, the rate g.

$g_{c_{v,t}}^*$ and g will generally differ. Suppose first that there is no retirement, i.e., $\lambda = 0$. Then we would have $g_{c_{v,t}}^* > g$. To take advantage of $r^* - \rho > g > 0$, the young (and in fact everybody) save, thus allowing individual consumption to rise at a rate higher than g. But only as long as alive. The relatively high consumption level by those who die is replaced by that of the newborn who have less financial wealth, hence lower consumption initially. This is what makes it possible that individual consumption can grow at a higher rate (until death) than per capita consumption (average consumption).
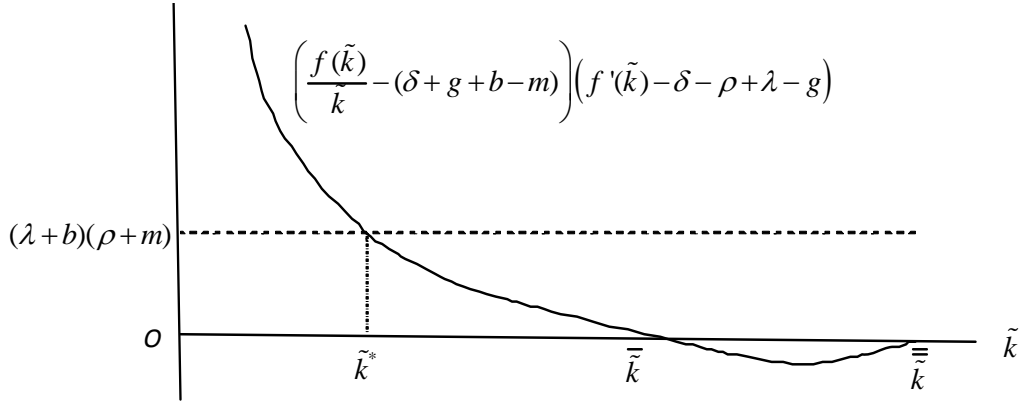
Figure 1.2:

With positive retirement ($\lambda > 0$), however, it is possible that $r^* - \rho < g$, cf. e) above. Then $g^*_{c_{v,t}} < g$. Only by an unlikely coincidence w.r.t. parameter values (a knife-edge case) will the two growth rates be equal.

g)   The interval (1.9) gives only a rough idea about how r^* depends on the birth rate, b. An unambiguous conclusion can be obtained in the following way. In steady state we have that $\tilde{k} = \tilde{k}^*$ and that the right-hand sides of (1.5) and (1.6) are equal to each other. By ordering we get

$$\left(\frac{f(\tilde{k})}{\tilde{k}} - (\delta + g + b - m) \right) \left[f'(\tilde{k}) - \delta - \rho + \lambda - g \right] = (\lambda + b)(\rho + m). \qquad (1.10)$$

Fig. 1.2 illustrates. A lower b makes the left-hand side of (1.10) larger for fixed $\tilde{k}$ and the right-hand side smaller. Thereby $\tilde{k}^*$ is raised and so $r^* = f'(\tilde{k}^*) - \delta$ becomes lower.

h)   The interval (1.9) also gives only a rough idea about how r^* depends on the technology growth rate, g. But (1.10) together with Fig. 1.2 gives a clear answer. A lower g makes the left-hand side of (1.10) larger and does not affect the right-hand side. Thereby $\tilde{k}^*$ is raised and so $r^* = f'(\tilde{k}^*) - \delta$ becomes lower.

Although the comparative-static method is not the same as a dynamic analysis of the effect of a gradual parameter change in historical time, it gives an idea about how lower b and lower g are likely to affect the rate of return.

2. Solution to Problem 2

The firm has the production function

$$y_t = K_t^\alpha L_t^{1-\alpha}, \qquad 0 < \alpha < 1. \qquad (*)$$

The real wage is an exogenous positive constant, w. The demand, y^d, for the firm's output is perceived by the firm as given by

$$y^d = p^{-\varepsilon}\frac{Y}{n}, \qquad \varepsilon > 1, \tag{**}$$

where p is the price set in advance by the firm relative to the general price level, n is the given large number of monopolistically competitive firms in the economy, and Y is the overall level of demand. Within the time horizon relevant for the analysis, Y is constant and the firm keeps p fixed. The capital depreciation rate, δ, satisfies $\delta > 0$.

There is a capital installation cost function $G(I_t)$ satisfying

$$G(0) = G'(0) = 0, \ G''(I) > 0. \tag{2.1}$$

a)   With $y_t = y^d$, (*) gives $L_t^{1-\alpha} = K_t^{-\alpha}y^d$ so that

$$L_t = (K_t^{-\alpha}y^d)^{\frac{1}{1-\alpha}} \equiv L(K_t, y^d).$$

b)   The decision problem is

$$\max_{(I_t)_{t=0}^{\infty}} V_0 = \int_0^{\infty} R_t e^{-rt}dt = \int_0^{\infty} \left[py^d - wL(K_t, y^d) - I_t - G(I_t)\right]e^{-rt}dt \quad \text{s.t.} \tag{2.2}$$

$$I_t \text{ free (i.e., no restriction on } I_t), \tag{2.3}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{2.4}$$

$$K_t \geq 0 \text{ for all } t. \tag{2.5}$$

By the Modigliani-Miller theorem we can separate the firm's investment-production problem from its financing problem. Given the solution in a) to the static part of the investment-production problem, namely the employment part, what is left of the investment-production problem is the intertemporal decision problem about capital investment, (2.2) - (2.5). In this problem the firm wants to choose an investment plan so as to maximize the value of the firm, that is, the present value of the expected stream of cash flows, R_t, taking not only labor costs and direct investment costs into account, but also the capital installation costs, represented by the strictly convex function $G(I_t)$. As there is no uncertainty, the discount rate r in (2.2) can be interpreted as the risk-free real interest rate. The control region, (2.3), says that also negative gross investment is possible (the firm may choose to dismantle and sell parts of its capital equipment). The link from control variable to change in the state variable is given in (2.4). Finally, the non-negativity constraint on the state variable in (2.5) is an implication of the definition of capital.

c)   The current-value Hamiltonian is:

$$H(K_t, I_t, q_t, t) \equiv py^d - wL(K_t, y^d) - I_t - G(I_t) + q_t(I_t - \delta K_t), \qquad (2.6)$$

where q_t is the adjoint variable associated with the dynamic constraint (2.4). The first-order conditions for an interior solution are for each $t \geq 0$ that

$$\frac{\partial H}{\partial I_t} = -1 - G'(I_t) + q_t = 0, \text{ i.e., } 1 + G'(I_t) = q_t, \qquad (2.7)$$

$$\frac{\partial H}{\partial K_t} = -wL_K(K_t, y^d) - q_t\delta = rq_t - \dot{q}_t, \qquad (2.8)$$

and the necessary transversality condition is

$$\lim_{t \to \infty} K_t q_t e^{-rt} = 0. \qquad (2.9)$$

d)   In view of $G'' \neq 0$, (2.7) defines optimal investment, I_t, as an implicit function of the adjoint variable q_t (the shadow price of installed capital):

$$I_t = \mathcal{M}(q_t), \qquad (2.10)$$

with properties

$$\mathcal{M}(1) = 0, \ \mathcal{M}'(q_t) = 1/G''(I_t) > 0. \qquad (2.11)$$

The first property follows from (2.7) together with (2.1) and the second follows by implicit differentiation in (2.7).

e)   The first differential equation entering the (K, q) dynamics is obtained by inserting (2.10) into (2.4):

$$\dot{K}_t = \mathcal{M}(q_t) - \delta K_t, \qquad K_0 > 0 \text{ given,}$$

where $\delta > 0$. We see that

$$\dot{K} \gtreqless 0 \text{ for } \mathcal{M}(q) \gtreqless \delta K, \quad \text{respectively.} \qquad (2.12)$$

As indicated in Fig. 2.1, the $\dot{K} = 0$ locus is upward-sloping in the (K, q) plane.

From a) follows that

$$L_K(K, y^d) = -\frac{\alpha}{1 - \alpha}\left(\frac{K}{y^d}\right)^{\frac{-1}{1-\alpha}}.$$

Substituting this into (2.8), we get the second differential equation in the system:

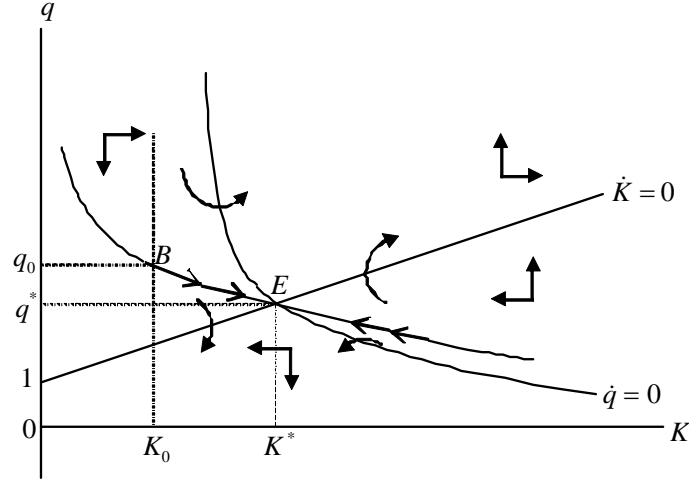$$\dot{q}_t = (r + \delta)q_t - w\frac{\alpha}{1 - \alpha}\left(\frac{K_t}{y^d}\right)^{\frac{-1}{1-\alpha}}.$$

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Figure 2.1:

We see that

$$\dot{q}_t \gtreqless 0 \text{ for } q_t \gtreqless \frac{w\alpha}{(r+\delta)(1-\alpha)} \left(\frac{K_t}{y^d} \right)^{\frac{-1}{1-\alpha}}. \tag{2.13}$$

As indicated in Fig. 2.1, the $\dot{q} = 0$ locus is downward-sloping.

There is a unique steady state, denoted E in Fig. 2.1, with coordinates K^* and q^*, respectively. Net investment is zero in the steady state. The directions of movement in the different regions of the phase diagram follow by (2.12) and (2.13), and are indicated by arrows in Fig. 2.1. The arrows taken together show that the steady state is a saddle point. As $r > 0$, the steady state satisfies the transversality condition, (2.9). Moreover, we have one predetermined variable, K, and one jump variable, q, the saddle path is not parallel to the jump-variable axis, and the diverging paths can be ruled out (they can be shown to violate the transversality condition). Hence the steady state is saddle-point stable.

For a given $K_0 > 0$, the path starting at the point B in Fig. 2.1 and moving along the saddle path towards the steady state is the unique solution, (K_t, q_t), to the firm's investment problem. This path is a solution because: (a) it satisfies the transversality condition; and (b) the Hamiltonian is jointly concave in (K, I). And the path is the $only$ solution because all other paths can be shown to violate the transversality condition. To the unique evolution of q_t corresponds a unique evolution of gross investment: $I_t = \mathcal{M}(q_t)$.

f)   The shift at time t_1 to $y^{d\prime} < y^d$ implies a downward shift in the $\dot{q} = 0$ locus as indicated in Fig. 2.2. The $\dot{K} = 0$ locus is not affected. At time t_1 the shadow price, q, jumps down to a level corresponding to the point B in Fig. 2.2. Then the pair (K_t, q_t)
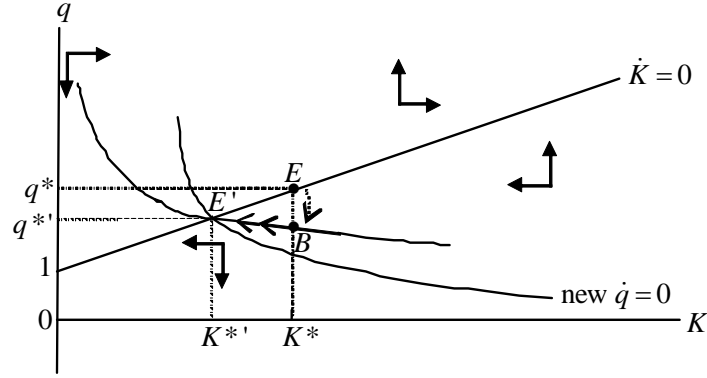
Figure 2.2:

moves gradually along the new saddle path towards the new steady state, E'. Gross investment, I_t, jumps down at time t_1 to a level below the depreciation of capital, so that net investment becomes negative. Then I_t gradually rises along with q_t towards the new steady state level, $I^{*'} = \mathcal{M}(q^{*'}) = \delta K^{*'}$, where gross investment exactly compensates for the depreciation of capital.

Note that $I^{*'} < I^* = \delta K^*$. So both on impact and in the long run is the effect on investment negative.

The economic intuition is as follows. The fall in demand implies less profitability of the marginal unit of installed capital. As a result the incentive to invest is diminished and gross investment will not even compensate for the depreciation of capital. Hence, the capital stock decreases gradually towards the new steady state level, $K^{*'} < K^*$. As the capital stock is gradually diminished, the marginal productivity of capital rises and so does therefore q_t.

g) Now it is instead the interest rate which at time t_1 shifts to a new constant level $r' > r$. Also this implies a downward shift in the $\dot{q} = 0$ locus, while the $\dot{K} = 0$ locus is not affected. So, qualitatively the implications for q, I, and K are the same as under f). The phase diagram in Fig. 2.2 is therefore still valid. Now it is a more heavy discounting of the future benefits of installed capital, rather than a fall in demand, which reduces the benefits of the marginal unit of installed capital.

h) In this case, at time t_1 the firm faces not only a fall in demand, but also more heavy discounting. So the $\dot{q} = 0$ locus is shifted further down and the negative effect on investment on impact as well as in the long run is greater than the isolated effects under f) and g), respectively.

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i)   A standard IS-LM model may be written (in standard notation):

$$
\begin{aligned}
Y &= Y^d = C(Y, r) + I(Y, r) + G, &\quad I_Y > 0, I_r < 0, \\
\frac{M}{P} &= L(Y, i), \\
r &= i - \pi,
\end{aligned}
$$

where we leave the properties of $C(\cdot)$ and $L(\cdot)$ unspecified and ignore that expected π may differ from actual π since these aspects are side issues here. What is important is the signs

$$
\frac{\partial I}{\partial r} = I_r < 0, \tag{2.14}
$$

$$
\frac{\partial I}{\partial Y} = I_Y > 0. \tag{2.15}
$$

As in our micro model, y^d is proportional to Y, the signs in (2.14) and (2.15) are the same as in our micro analysis, whether we think of the impact effect or the longer-run effect. This provides some microfoundation for the investment function in the IS-LM model.

3. Solution to Problem 3

a)   A *Beveridge curve* is a graphical relationship between the unemployment rate and the job vacancy rate (the number of vacant jobs expressed as a proportion of the labor force). The curve typically has the unemployment rate on the horizontal axis and the vacancy rate on the vertical axis.

The Beveridge curve is generally negatively sloped. In a boom, unemployment is low and vacancies plenty because recruitment is difficult, as few workers are searching for a job. In a slump, unemployment is high and the vacancy rate low because recruitment is easy, as many workers are searching for a job. In this way, the economy's position on the downward sloping Beveridge curve can be interpreted as reflecting the state of the business cycle.

An outward shift of the Beveridge curve is a sign of reduced matching efficiency in the labor market. Such a *mismatch* phenomenon can be due to fast technological and structural change. Firms in the new industries have vacant jobs but it is hard to find appropriate workers. Increased mismatch can also result from a prolonged economic slump leading to long-term unemployment with negative social consequences (de-qualification, discouragement).

b)   A believer of the Schumpeterian story would expect "total separations", "quits", and "hiring" to rise in the recession because the recession is seen as reflecting high fric-

tional unemployment as workers move from obsolete industries to blossoming industries offering new types of jobs with attractive wages.

A believer of the Keynesian story would expect "layoffs and discharges" to rise because firms generally need fewer workers to satisfy the slack demand. A believer of the Keynesian story would expect "hiring" to fall in the recession because firms generally need fewer new workers in a situation with slack demand. Finally, a believer of the Keynesian story would expect "quits" to fall because workers are hesitant to quit as they perceive that vacant alternative jobs are scarce.

c)   The data on labor market flows in the U.S. published by the Bureau of Labor Statistics tells us the following:

- During the recession in 2001 (associated with the crash of the dot-com boom) as well as during the Great Recession 2008-2009, "total separations", "quits", and "hiring" were systematically *falling*. These three features are exactly the opposite of what the believer of the Schumpeterian story would expect.

- The observed fall in "quits" and "hiring" during the two recessions are in line with what the believer of the Keynesian story would expect. Moreover, the data also shows that "layoffs and discharges" rose considerably during the Great Recession, again in line with the Keynesian story about the nature of this recession. In the relatively short 2001 recession "layoffs and discharges" showed no clear tendency.

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