Chapter 11

Applications of the Ramsey model

The Ramsey representative agent framework has, rightly or wrongly, been a workhorse for the study of many macroeconomic issues. Among these are public finance themes and themes relating to endogenous productivity growth. In this chapter we consider issues within these two themes. Section 11.1 deals with a market economy with a public sector. The focus is on general equilibrium effects of government spending and taxation, including effects of shifts in fiscal policy, both anticipated and unanticipated shifts. In Section 11.2 we set up and analyze a model of technology growth based on learning by investing. The analysis leads to a characterization of a "first-best policy".

11.1 Market economy with a public sector

In this section we extend the Ramsey model of a competitive market economy by adding a government sector that spends on goods and services, makes transfers to the private sector, and levies taxes.

Subsection 11.1.1 considers the effect of government spending on goods and services, assuming a balanced budget where all taxes are lump sum. The issue what is really meant by one-off shocks in a perfect foresight model is addressed, including how to model the effects of such shocks. In subsections 11.1.2 and 11.1.3 we consider income taxation and how the economy responds to the arrival of new information about future fiscal policy. Finally, subsection 11.1.4 introduces financing by temporary budget deficits. In view of the Ramsey model being a representative agent model, it is not surprising that Ricardian equivalence will hold in the model.

11.1.1 Public consumption financed by lump-sum taxes

The representative household (or family dynasty) has $L_t = L_0 e^{nt}$ members each of which supplies one unit of labor inelastically per time unit, $n \geq 0$. The household's preferences can be represented by a time separable utility function

$$\int_0^\infty \tilde{u}(c_t, G_t) L_t e^{-\rho t} dt,$$

where $c_t \equiv C_t/L_t$ is consumption per family member and G_t is public consumption in the form of a service delivered by the government, while ρ is the rate of time preference. We assume, for simplicity, that the instantaneous utility function is additive: $\tilde{u}(c, G) = u(c) + v(G)$, where $u' > 0, u'' < 0$, i.e., there is positive but diminishing marginal utility of private consumption; the properties of the utility function v are immaterial for the questions to be studied (but hopefully $v' > 0$). The public service might consist in making a non-rival good, say "law and order" or TV-transmitted theatre, available for the households free of charge.

Throughout this section the government budget is always balanced. In the present subsection the government spending, G_t, is financed by a per capita lump-sum tax, τ_t, so that

$$\tau_t L_t = G_t. \tag{11.1}$$

To allow for balanced growth under technological progress we assume that u is a CRRA function. Thus, the criterion function of the representative household can be written

$$U_0 = \int_0^\infty \left(\frac{c_t^{1-\theta}}{1-\theta} + v(G_t) \right) e^{-(\rho - n)t} dt, \tag{11.2}$$

where $\theta > 0$ is the constant (absolute) elasticity of marginal utility of private consumption.

As usual, let the real interest rate and the real wage be denoted r_t and w_t, respectively. The household's dynamic book-keeping equation reads

$$\dot{a}_t = (r_t - n)a_t + w_t - \tau_t - c_t, \qquad a_0 \text{ given}, \tag{11.3}$$

where a_t is per capita financial wealth. The financial wealth is held in claims of a form similar to a variable-rate deposit in a bank. Hence, at any point in time a_t is historically determined and independent of the current and future interest rates. The No-Ponzi-Game condition (solvency condition) is

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n) ds} \geq 0. \tag{NPG}$$

We see from (11.2) that leisure does not enter the instantaneous utility function. So per capita labor supply is exogenous. We fix its value to be one unit of labor per time unit, as is indicated by (11.3).

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In view of the additive instantaneous utility function in (11.2), marginal utility of private consumption is not affected by G_t. The Keynes-Ramsey rule resulting from the household's optimization will therefore be as if there were no government sector:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho).$$

The transversality condition of the household is that (NPG) holds with strict equality, i.e.,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0.$$

GDP is produced through an aggregate neoclassical production function with CRS:

$$Y_t = F(K_t^d, \mathcal{T}_t L_t^d),$$

where K_t^d and L_t^d are inputs of capital and labor, respectively, and $\mathcal{T}_t$ is the technology level, assumed to grow at an exogenous and constant rate $g \geq 0$. For simplicity we assume that F satisfies the Inada conditions. It is further assumed that in the production of G_t the same technology (production function) is applied as in the production of the other components of GDP; thereby the same unit production costs are involved. A possible role of G_t for productivity is ignored (so we should not interpret G_t as related to such things as infrastructure, health, education, or research).

All capital in the economy is assumed to belong to the private sector. The economy is closed. In accordance with the standard Ramsey model, there is perfect competition in all markets. Hence there is market clearing so that $K_t^d = K_t$ and $L_t^d = L_t$ for all t.

General equilibrium and dynamics

The increase in the capital stock, K, per time unit equals aggregate gross saving:

$$\dot{K}_t = Y_t - C_t - G_t - \delta K_t = F(K_t, \mathcal{T}_t L_t) - c_t L_t - G_t - \delta K_t, \qquad K_0 > 0 \text{ given.} \quad (11.4)$$

We assume G_t is proportional to the work force measured in efficiency units, that is

$$G_t = \tilde{\gamma}\mathcal{T}_t L_t, \quad (11.5)$$

where the size of $\tilde{\gamma} \geq 0$ is decided by the government. The balanced budget (11.1) now implies that the per capita lump-sum tax grows at the same rate as technology:

$$\tau_t = G_t/L_t = \tilde{\gamma}\mathcal{T}_t = \tilde{\gamma}\mathcal{T}_0 e^{gt} = \tau_0 e^{gt}. \quad (11.6)$$

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Defining $\tilde{k}_t \equiv K_t/(\mathcal{T}_t L_t) \equiv k_t/\mathcal{T}_t$ and $\tilde{c}_t \equiv C_t/(\mathcal{T}_t L_t) \equiv c_t/\mathcal{T}_t$, the dynamic aggregate resource constraint (11.4) can be written

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - \tilde{\gamma} - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{11.7}$$

where f is the production function in intensive form, $f' > 0, f'' < 0$. As F satisfies the Inada conditions, we have

$$f(0) = 0, \qquad \lim_{\tilde{k}\to 0} f'(\tilde{k}) = \infty, \qquad \lim_{\tilde{k}\to\infty} f'(\tilde{k}) = 0.$$

As usual, by the golden-rule capital intensity, $\tilde{k}_{GR}$, we mean that capital intensity which maximizes sustainable consumption per unit of effective labor, $\tilde{c} + \tilde{\gamma}$. By setting the left-hand side of (11.7) to zero, eliminating the time indices on the right-hand side, and rearranging, we get $\tilde{c} + \tilde{\gamma} = f(\tilde{k}) - (\delta + g + n)\tilde{k} \equiv c(\tilde{k})$. In view of the Inada conditions, the problem $\max_{\tilde{k}} c(\tilde{k})$ has a unique solution, $\tilde{k} > 0$, characterized by the condition $f'(\tilde{k}) = \delta + g + n$. This $\tilde{k}$ is, by definition, $\tilde{k}_{GR}$.

In general equilibrium the real interest rate, r_t, equals $f'(\tilde{k}_t) - \delta$. Expressed in terms of $\tilde{c}$, the Keynes-Ramsey rule thus becomes

$$\dot{\tilde{c}}_t = \frac{1}{\theta}\left[f'(\tilde{k}_t) - \delta - \rho - \theta g\right]\tilde{c}_t. \tag{11.8}$$

Moreover, we have $a_t = k_t \equiv \tilde{k}_t T_t = \tilde{k}_t T_0 e^{gt}$, and so the transversality condition of the representative household can be written

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left(f'(\tilde{k}_s) - \delta - n - g\right) ds} = 0. \tag{11.9}$$

The phase diagram of the dynamic system (11.7) - (11.8) is shown in Fig. 11.1 where, to begin with, the $\dot{\tilde{k}} = 0$ locus is represented by the stippled inverse U curve. Apart from a vertical downward shift of the $\dot{\tilde{k}} = 0$ locus, when we have $\tilde{\gamma} > 0$ instead of $\tilde{\gamma} = 0$, the phase diagram is similar to that of the Ramsey model without government. Although the per capita lump-sum tax is not visible in the reduced form of the model consisting of (11.7), (11.8), and (11.9), it is indirectly present because it ensures that for all $t \geq 0$, the $\tilde{c}_t$ and $\dot{\tilde{k}}_t$ appearing in (11.7) represent exactly the consumption demand and net saving coming from the households' intertemporal budget constraint (which depends on the lump-sum tax, cf. (11.11). Otherwise, equilibrium would not be maintained.

We assume $\tilde{\gamma}$ is of "moderate size" compared to the productive capacity of the economy so as to not rule out the existence of a steady state. Moreover, to

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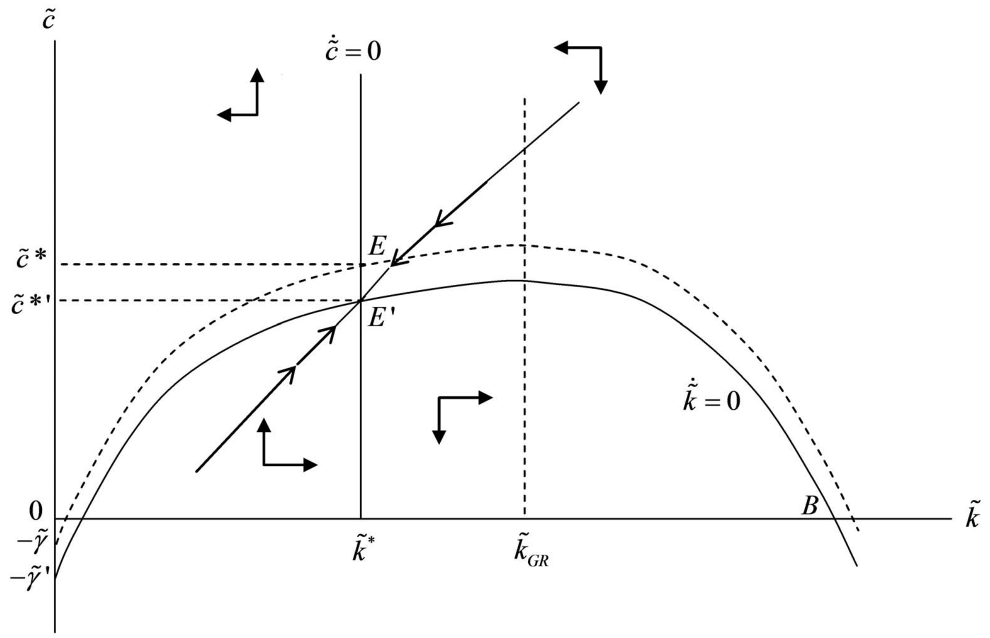

Figure 11.1: Phase portrait of an unanticipated permanent increase in government spending from $\tilde{\gamma}$ to $\tilde{\gamma}' > \tilde{\gamma}$.

guarantee bounded discounted utility and existence of general equilibrium, we impose the parameter restriction

$$\rho - n > (1 - \theta)g. \tag{A1}$$

How to model effects of unanticipated policy shifts

In a perfect foresight model, as the present one, agents' expectations and actions never incorporate that unanticipated events, "shocks", may arrive. That is, if a shock occurs in historical time, it must be treated as a complete surprise, a one-off shock not expected to be replicated in any sense.

Suppose that up until time $t_0 > 0$ government spending maintains the given ratio $G_t/(\mathcal{T}_t L_t) = \tilde{\gamma}$. Suppose further that before time t_0, the households expected this state of affairs to continue forever. But, unexpectedly, at time t_0 there is a shift to a higher constant spending ratio, $\tilde{\gamma}'$, which is maintained for a long time.

We assume that the upward shift in public spending goes hand in hand with higher lump-sum taxes so as to maintain a balanced budget. Thereby the after-tax human wealth of the household is at time t_0 immediately reduced. As the households are now less wealthy, private consumption immediately drops.

Mathematically, the time path of c_t will therefore have a discontinuity at $t = t_0$. To fix ideas, we will generally consider *control* variables, e.g., consumption,

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to be *right-continuous* functions of time in such cases. This means that $c_{t_0} = \lim_{t \to t_0^+} c_t$. Likewise, at such points of discontinuity of the control variable the "time derivative" of the *state* variable a in (11.3) is generally not well-defined without an amendment. In line with the right-continuity of the control variable, we define the time derivative of a state variable at a point of discontinuity of the control variable as the *right-hand time derivative*, i.e., $\dot{a}_{t_0} = \lim_{t \to t_0^+}(a_t - a_{t_0})/(t - t_0)$.[1] We say that the control variable has a *jump* at time t_0, we call the point where this jump occurs a *switch point*, and we say that the state variable, which remains a continuous function of t, has a *kink* at time t_0.

In line with this, control variables are called *jump variables* or *forward-looking variables.* The latter name comes from the notion that a decision variable can immediately shift to another value if new information arrives. In contrast, a state variable is said to be *pre-determined* because its value is an outcome of the past and it cannot jump.

An unanticipated permanent shift in government spending Returning to our specific example, suppose that the economy has been in steady state for $t < t_0$. Then, unexpectedly, the new spending policy $\tilde{\gamma}' > \tilde{\gamma}$ is introduced, followed by an increase in taxation so as to maintain a balanced budget. Let the households rightly expect this new policy to be maintained forever. As a consequence, the $\dot{\tilde{k}} = 0$ locus in Fig. 11.1 is shifted downwards while the $\dot{\tilde{c}} = 0$ locus remains where it is. It follows that $\tilde{k}$ stays unchanged at its old steady-state level, $\tilde{k}^*$, while $\tilde{c}$ jumps down to the new steady-state value, $\tilde{c}^{*\prime}$. There is immediate crowding out of private consumption to the exact extent of the rise in public consumption.[2]

To understand the mechanism, note that Per capita consumption of the household is

$$c_t = \beta_t(a_t + h_t), \tag{11.10}$$

where h_t is the after-tax human wealth per family member and is given by

$$h_t = \int_t^\infty (w_s - \tau_s) e^{-\int_t^s (r_z - n)dz} ds, \tag{11.11}$$

and β_t is the propensity to consume out of wealth,

$$\beta_t = \frac{1}{\int_t^\infty e^{\int_t^s (\frac{(1-\theta)r_z - \rho}{\theta} + n)dz} ds}, \tag{11.12}$$

[1] While these conventions help to fix ideas, they are mathematically inconsequential. Indeed, the value of the consumption intensity at each isolated point of discontinuity will affect neither the utility integral of the household nor the value of the state variable, a.

[2] The conclusion is modified, of course, if G_t encompasses public investments and if these have an impact on the productivity of the private sector.

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as derived in the previous chapter. The upward shift in public spending is accompanied by higher lump-sum taxes, $\tau_t' = \tilde{\gamma}' L_t$, forever, implying that h_t is reduced, which in turn reduces consumption.

Had the unanticipated shift in public spending been *downward*, say from $\tilde{\gamma}'$ to $\tilde{\gamma}$, the effect would be an *upward* jump in consumption but no change in $\tilde{k}$, that is, a jump E' to E in Fig. 11.1.

Many kinds of disturbances of a steady state will result in a *gradual* adjustment process, either to a new steady state or back to the original steady state. It is otherwise in this example where there is an *immediate jump* to a new steady state.

11.1.2 Income taxation

We now replace the assumed lump-sum taxation by income taxation of different kinds. In addition, we introduce lump-sum income transfers to the households.

Taxation of labor income

Consider a tax on wage income at the constant rate τ_w, $0 < \tau_w < 1$. Since labor supply is exogenous, it is unaffected by the wage income tax. While (11.7) is still the dynamic resource constraint of the economy, the household's dynamic book-keeping equation now reads

$$\dot{a}_t = (r_t - n)a_t + (1 - \tau_w)w_t + x_t - c_t, \qquad a_0 \text{ given,}$$

where x_t is the per capita lump-sum transfers at time t. Maintaining the assumption of a balanced budget, the tax revenue at every t exactly covers government spending on goods and services and the lump-sum transfers to the private sector. This means that

$$\tau_w w_t L_t = G_t + x_t L_t \qquad \text{for all } t \geq 0.$$

As G_t and τ_w are given, the interpretation is that for all $t \geq 0$, transfers adjust so as to balance the budget. This requires that $x_t = \tau_w w_t - G_t/L_t = \tau_w w_t - \tilde{\gamma}\mathcal{T}_t$, for all $t \geq 0$; if x_t need be negative to satisfy this equation, so be it. Then $-x_t$ would act as a positive lump-sum tax.

Disposable income at time t is

$$(1 - \tau_w)w_t + x_t = w_t - \tilde{\gamma}\mathcal{T}_t,$$

and human wealth at time t per member of the representative household is thus

$$h_t = \int_t^\infty \left[(1 - \tau_w)w_s + x_s\right] e^{-\int_t^s (r_z - n)dz} ds = \int_t^\infty (w_s - \tilde{\gamma}\mathcal{T}_s) e^{-\int_t^s (r_z - n)dz} ds. \tag{11.13}$$

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Owing to the given $\tilde{\gamma}$, a shift in the value of τ_w is immediately compensated by an adjustment of the path of transfers in the same direction so as to maintain a balanced budget. Neither disposable income nor h_t is affected. So the shift in τ_w leaves the determinants of per capita consumption unaffected. As also disposable income is unaffected, it follows that private saving is unaffected. This is why τ_w nowhere enters the model in its reduced form, consisting of (11.7), (11.8), and (11.9). The phase diagram for the economy with labor income taxation is completely identical to that in Fig. 11.1 where there is no tax on labor income. The evolution of the economy is independent of the size of τ_w (if the model were extended with endogenous labor supply, the result would generally be different). The intuitive explanation is that the three conditions: (a) inelastic labor supply, (b) a balanced budget,[3] and (c) a given path for G_t, imply that a labor income tax affects neither the marginal trade-offs (consumption versus saving and working versus enjoying leisure) nor the intertemporal budget constraint of the household.

Taxation of capital income

It is different when it comes to a tax on capital income because saving in the Ramsey model responds to incentives. Consider a constant capital income tax at the rate τ_r, $0 < \tau_r < 1$. The household's dynamic budget identity becomes

$$\dot{a}_t = [(1 - \tau_r)r_t - n]\, a_t + w_t + x_t - c_t, \qquad a_0 \text{ given},$$

where, if $a_t < 0$, the tax acts as a rebate. As above, x_t is a per capita lump-sum transfer. In view of a balanced budget, we have at the aggregate level

$$G_t + x_t L_t = \tau_r r_t K_t.$$

As G_t and τ_r are given, the interpretation is that for all $t \geq 0$, transfers adjust so as to balance the budget. This requires that $x_t = \tau_r r_t k_t - G_t/L_t = \tau_r r_t k_t - \tilde{\gamma}\mathcal{T}_t$.

The No-Ponzi-Game condition is now

$$\lim_{t\to\infty} a_t e^{-\int_0^t [(1-\tau_r)r_s - n]ds} \geq 0,$$

and the Keynes-Ramsey rule becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[(1 - \tau_r)r_t - \rho].$$

[3] In fact, as we shall see in Section 11.1.4, the key point is not that, to fix ideas, we have assumed the budget is balanced for every t. It is enough that the government satisfies its intertemporal budget constraint.

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In general equilibrium we have

$$\dot{\tilde{c}}_t = \frac{1}{\theta}\left[(1-\tau_r)(f'(\tilde{k}_t)-\delta)-\rho-\theta g\right]\tilde{c}_t. \tag{11.14}$$

The differential equation for $\tilde{k}$ is still (11.7).

In steady state we get $(f'(\tilde{k}^*)-\delta)(1-\tau_r)=\rho+\theta g$, that is,

$$f'(\tilde{k}^*)-\delta = \frac{\rho+\theta g}{1-\tau_r} > \rho+\theta g > g+n,$$

where the last inequality comes from the parameter condition (A1). Because $f''<0$, $\tilde{k}^*$ is lower than if $\tau_r=0$. Consequently, in the long run consumption is lower as well.[4] The resulting resource allocation is not Pareto optimal. There exist an alternative technically feasible resource allocation that makes everyone in society better off. This is because the capital income tax implies a wedge between the marginal rate of transformation over time in production, $f'(\tilde{k}_t)-\delta$, and the marginal rate of transformation over time to which consumers adapt, $(1-\tau_r)(f'(\tilde{k}_t)-\delta)$.

11.1.3 Effects of shifts in the capital income tax rate

We shall study effects of a rise in the tax on capital income. The effects depend on whether the change is anticipated in advance or not and whether the change is permanent or only temporary. So there are four cases to consider.

(i) Unanticipated permanent shift in τ_r

Until time t_0 the economy has been in steady state with a tax-transfer scheme based on some given constant tax rate, τ_r, on capital income. At time t_0, unexpectedly, the government introduces a new tax-transfer scheme, involving a higher constant tax rate, τ_r', on capital income, i.e., $0<\tau_r<\tau_r'<1$. The path of spending on goods and services remains unchanged, i.e., $G_t=\tilde{\gamma}\mathcal{T}_tL_t$ for all $t\geq 0$. The lump-sum transfers, x_t, are raised so as to maintain a balanced budget. We assume it is credibly announced that the new tax-transfer scheme will be adhered to forever. So households expect the real after-tax interest rate (rate of return on saving) to be $(1-\tau_r')r_t$ for all $t\geq t_0$.

For $t<t_0$ the dynamics are governed by (11.7) and (11.14) with $0<\tau_r<1$. The corresponding steady state, E, has $\tilde{k}=\tilde{k}^*$ and $\tilde{c}=\tilde{c}^*$ as indicated in the

[4] In the Diamond OLG model a capital income tax, which finances lump-sum transfers to the old generation, has an ambiguous effect on capital accumulation, depending on whether $\theta<1$ or $\theta>1$, cf. Exercise 5.?? in Chapter 5.

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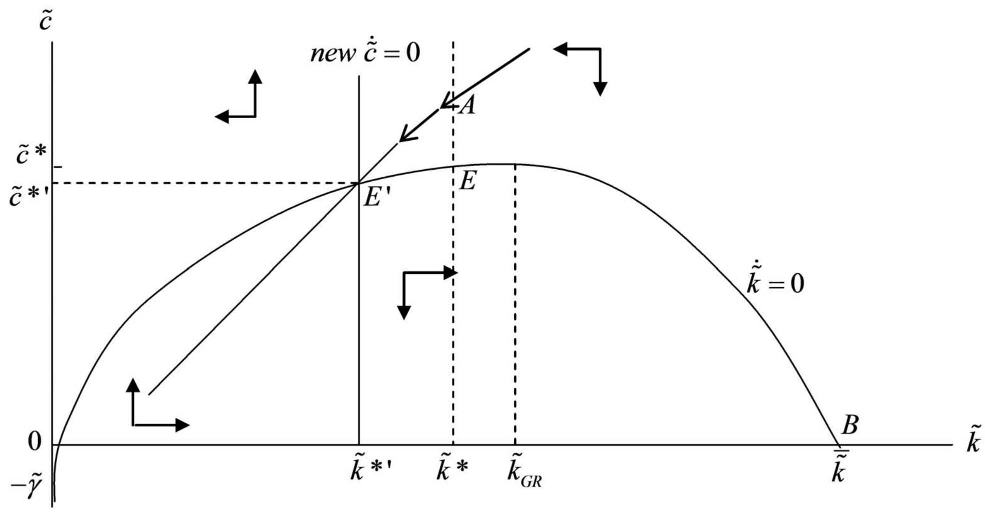

Figure 11.2: Phase portrait of an unanticipated permanent rise in τ_r.

phase diagram in Fig. 11.2. The new tax-transfer scheme ruling after time t_0 shifts the steady state point to E' with $\tilde{k} = \tilde{k}^{*\prime}$ and $\tilde{c} = \tilde{c}^{*\prime}$. The new $\dot{\tilde{c}} = 0$ line and the new saddle path are to the left of the old, i.e., $\tilde{k}^{*\prime} < \tilde{k}^{*}$. Until time t_0 the economy is at the point E. Immediately after the shift in the tax on capital income, equilibrium requires that the economy is on the new saddle path. So there will be a jump from point E to point A in Fig. 11.2.

This upward jump in consumption is intuitively explained the following way. We know that individual consumption immediately after the policy shock satisfies

$$\begin{aligned} c_{t_0} &= \beta_{t_0}(a_{t_0} + h_{t_0}), \qquad \text{where} \\ h_{t_0} &= \int_{t_0}^{\infty} (w_t + \tau_r' r_t k_t - \tilde{\gamma} T_t) e^{-\int_{t_0}^{t}((1-\tau_r')r_z - n)dz} dt, \qquad \text{and} \\ \beta_{t_0} &= \frac{1}{\int_{t_0}^{\infty} e^{\int_{t_0}^{t}(\frac{(1-\theta)(1-\tau_r')r_z - \rho}{\theta} + n)dz} dt}. \end{aligned} \tag{11.15}$$

Two effects are present. First, both the higher transfers and the lower after-tax rate of return after time t_0 contribute to a higher h_{t_0}; there is thereby a positive wealth effect on current consumption through a higher h_{t_0}. Second, the propensity to consume, β_{t_0}, will generally be affected. If $\theta < 1$, the reduction in the after-tax rate of return will have a positive effect on β_{t_0}. The positive effect on β_{t_0} when $\theta < 1$ reflects that the positive substitution effect on c_{t_0} of a lower after-tax rate of return dominates the negative income effect. If instead $\theta > 1$, the positive substitution effect on c_{t_0} is dominated by the negative income effect. Whatever happens to β_{t_0}, however, the phase diagram shows that in general equilibrium there will necessarily be an *upward* jump in c_{t_0}. We get this result even if θ is much higher than 1. The explanation lies in the assumption that all the extra tax revenue obtained by the rise in τ_r is immediately transferred back to the households lump sum, thereby strengthening the positive wealth effect on current consumption through the lower discount rate implied by $(1-\tau_r')r_z < (1-\tau_r)r_z$.

In response to the rise in τ_r, we thus have $\tilde{c}_{t_0} > f(\tilde{k}_{t_0}) - (\delta+g+n)\tilde{k}_{t_0}$, implying that saving is too low to sustain $\tilde{k}$, which thus begins to fall. This results in lower real wages and higher before-tax interest rates, that is two *negative* feedbacks on human wealth. Could these feedbacks not fully offset the initial tendency for (after-tax) human wealth to rise? The answer is no, see Box 11.1.

As indicated by the arrows in Fig. 11.2, the economy moves along the new saddle path towards the new steady state E'. Because $\tilde{k}$ is lower in the new steady state than in the old, so is $\tilde{c}$. The evolution of the technology level, T, is by assumption exogenous; thus, also actual per capita consumption, $c \equiv \tilde{c}^* T$, is lower in the new steady state.

Box 11.1. A mitigating feedback can not instantaneously fully offset the force that activates it.

Can the story told by Fig. 11.2 be true? Can it be true that the net effect of the higher tax on capital income is an upward jump in consumption at time t_0 as indicated in Fig. 11.2? Such a jump means that $\tilde{c}_{t_0} > f(\tilde{k}_{t_0}) - (\delta + g + n)\tilde{k}_{t_0}$ and the resulting reduced saving will make the future k lower than otherwise and thereby make expected future real wages lower and expected future before-tax interest rates higher. Both feedbacks partly counteract the initial upward shift in human wealth due to higher transfers and a lower effective discount rate that were the direct result of the rise in τ_w. Could the two mentioned counteracting feedbacks fully offset the initial tendency for (after-tax) human wealth, and therefore current consumption, to rise?

The phase diagram says no. But what is the intuition? That the two feedbacks can not fully offset (or even reverse) the tendency for (after-tax) human wealth to rise at time t_0 is explained by the fact that if they could, then the two feedbacks would not be there in the first place. We cannot at the same time have both a rise in the human wealth that triggers higher consumption (and thereby lower saving and investment in the economy) and a neutralization, or a complete reversal, of this rise in the human wealth caused by the higher consumption. The two feedbacks can only partly offset the initial tendency for human wealth to rise.

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Instead of all the extra tax revenue obtained being transferred back lump sum to the households, we may alternatively assume that a major part of it is used to finance a rise in government consumption to the level $G_t' = \tilde{\gamma}' T_t L_t$, where $\tilde{\gamma}' > \tilde{\gamma}$.[5] In addition to the leftward shift of the $\dot{\tilde{c}} = 0$ locus this will result in a downward shift of the $\dot{\tilde{k}} = 0$ locus. The phase diagram would look like a convex combination of Fig. 11.1 and Fig. 11.2. Then it is possible that the jump in consumption at time t_0 becomes downward instead of upward.

Returning to the case where the extra tax revenue is fully transferred, the next subsection splits the change in taxation policy into two events.

(ii) Anticipated permanent shift in τ_r

Until time t_0 the economy has been in steady state with a tax-transfer scheme based on some given constant tax rate, τ_r, on capital income. At time t_0, unexpectedly, the government announces that a new tax-transfer policy with $\tau_r' > \tau_r$ is to be implemented at time $t_1 > t_0$. We assume people believe in this announcement and that the new policy is implemented at time t_1 as announced. The shock to the economy is now not the event of a higher tax being implemented at time t_1; this event is expected after time t_0. The shock occurs at time t_0 in the form of the unexpected announcement. The path of spending on goods and services remains unchanged throughout, i.e., $G_t = \tilde{\gamma} \mathcal{T}_t L_t$ for all $t \geq 0$.

The phase diagram in Fig. 11.3 illustrates the evolution of the economy for $t \geq t_0$. There are two time intervals to consider. For $t \in [t_1, \infty)$, the dynamics are governed by (11.7) and (11.14) with τ_r replaced by τ_r', starting from whatever value obtained by $\tilde{k}$ at time t_1.

In the time interval $[t_0, t_1)$, however, the "old dynamics", with the lower tax rate, τ_r, in a sense still hold. Yet the path the economy follows immediately after time t_0 is different from what it would be without the information that capital income will be taxed heavily from time t_1, where also transfers will become higher. Indeed, the expectation of a lower after-tax interest rate until time t_1, combined with higher transfers from time t_1 implies higher present value of future labor and transfer income. Already at time t_0 this induces an upward jump in consumption to the point C in Fig. 11.3 because people feel more wealthy.

Since the low τ_r rules until time t_1, the point C is below the point A, which is the same as that in Fig. 11.2. How far below? The answer follows from the fact that there cannot be an *expected* discontinuity of marginal utility at time t_1, since that would contradict the preference for consumption smoothing over time

[5] It is understood that also $\tilde{\gamma}'$ is not larger than what allows a steady state to exist. Moreover, the government budget is still balanced for all t so that any temporary surplus or shortage of tax revenue, $\tau_r' r_t K_t - G_t'$, is immediately transferred or collected lump-sum.

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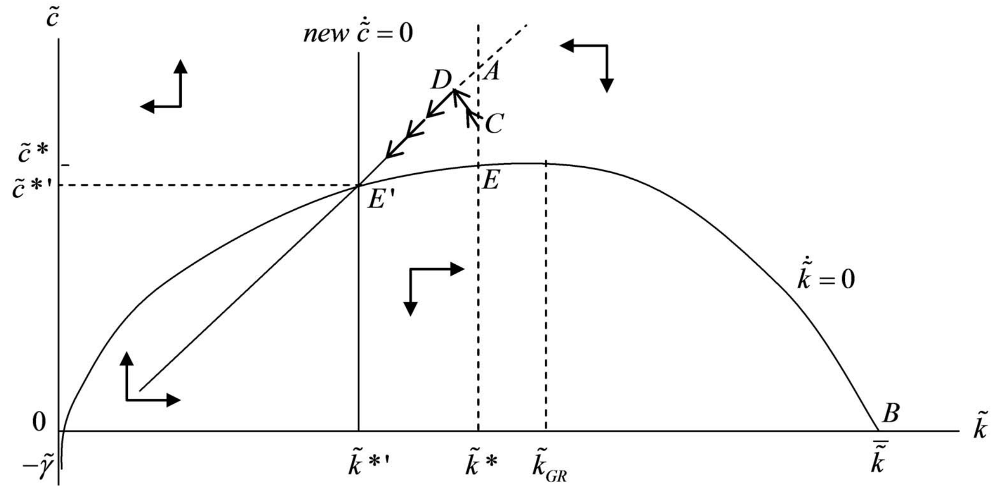

Figure 11.3: Phase portrait of an anticipated permanent rise in τ_r.

implied by $u''(c) < 0$ (strict concavity of the instantaneous utility function) and reflected in the Keynes-Ramsey rule. To put it differently: the shift to τ_r' does not occur immediately, as in (11.15), but in the future, and as long as the shift is known to occur at a given time in the future. The shift, when it takes place, namely at the announced time t_1, will not trigger a jump in human wealth, h_{t_1}.[6] Hence, at time t_1, there will be no jump in consumption, c_{t_1}.

The intuitive background for this is that a consumer will never *plan* a jump in consumption. To see this, consider a consumption path in the time interval (t_0, t_2), where $t_2 > t_1$. Suppose there is a discontinuity in c_t at time t_1. In view of the strict concavity of the utility function, there would then be gains to be obtained by smoothing out consumption. Recalling the optimality condition $u'(c_{t_1}) = \lambda_{t_1}$, we could also say that along an optimal path there can be no *expected* discontinuity in the shadow price of financial wealth, λ_{t_1}. This is analogue to the fact that in an asset market, arbitrage rules out the existence of a generally expected jump in the price of the asset to occur at some future time t_1. If we imagine the expected jump is upward, an infinite positive rate of return could be obtained by buying the asset immediately before the jump. This generates excess demand of the asset before time t_1 and drives its price up in advance thus *preventing* an expected upward jump at time t_1. And if we on the other hand imagine the expected jump is downward, an infinite negative rate of return could be avoided by selling the asset immediately before the jump. This generates ex-

[6]Replace t_0 in the formula for human wealth in (11.15) by some $t \in (t_0, t_1)$, and consider h_t as the sum of the integrals from t to t_1 and from t_1 to ∞, respectively, and let then t approach t_1 from below.

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cess supply of the asset before time t_1 and drives its price down in advance thus preventing an expected downward jump at t_1.

To avoid existence of an expected discontinuity in consumption, the point C on the vertical line $\tilde{k} = \tilde{k}^*$ in Fig. 11.3 must be such that, following the "old dynamics", it takes exactly $t_1 - t_0$ time units to reach the new saddle path. This dictates a unique position of the point C between E and A. If C were at a lower position, the journey to the saddle path would take longer than $t_1 - t_0$. And if C were at a higher position, the journey would not take as long as $t_1 - t_0$.

Immediately after time t_0, $\tilde{k}$ will be decreasing (because saving is smaller than what is required to sustain a constant $\tilde{k}$); and $\tilde{c}$ will be *increasing* in view of the Keynes-Ramsey rule, since the rate of return on saving is above $\rho + \theta g$ as long as $\tilde{k} < \tilde{k}^*$ and τ_r low. Precisely at time t_1 the economy reaches the new saddle path, the high taxation of capital income begins, and the after-tax rate of return becomes lower than $\rho + \theta g$. Hence, per-capita consumption begins to fall and the economy gradually approaches the new steady state E'.

This analysis illustrates that when economic agents' behavior depend on forward-looking expectations, a credible announcement of a future change in policy has an effect already before the new policy is implemented. Such effects are known as *announcement effects* or *anticipation effects*.

(iii) Unanticipated temporary shift in τ_r

Once again we change the scenario. The economy with low capital taxation has been in steady state up until time t_0. Then a new tax-transfer scheme is unexpectedly introduced. At the same time it is credibly announced that the high taxes on capital income and the corresponding transfers will cease at time $t_1 > t_0$. The path of spending on goods and services remains unchanged throughout, i.e., $G_t = \tilde{\gamma}\mathcal{T}_t L_t$ for all $t \geq 0$.

The phase diagram in Fig. 11.4 illustrates the evolution of the economy for $t \geq t_0$. For $t \geq t_1$, the dynamics are governed by (11.7) and (11.14), again with the old τ_r, starting from whatever value obtained by $\tilde{k}$ at time t_1.

In the time interval $[t_0, t_1)$ the "new, temporary dynamics" with the high τ_r' and high transfers hold sway. Yet the path that the economy takes immediately after time t_0 is different from what it would have been without the information that the new tax-transfers scheme is only temporary. Indeed, the expectation of a shift to a higher after-tax rate of return and cease of high transfers as of time t_1 implies lower present value of expected future labor and transfer earnings than without this information. Hence, the upward jump in consumption at time t_0 is smaller than in Fig. 11.2. How much smaller? Again, the answer follows from the fact that there can not be an *expected* discontinuity of marginal utility at time t_1, since that would violate the principle of smoothing of planned consumption.

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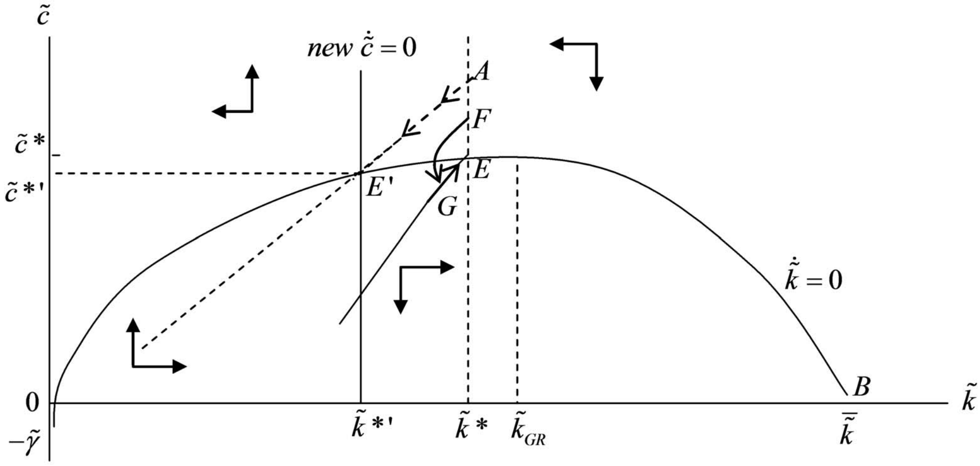

Figure 11.4: Phase portrait of an unanticipated temporary rise in τ_r.

Thus the point F on the vertical line $\tilde{k} = \tilde{k}^*$ in Fig. 11.4 must be such that, following the "new, temporary dynamics", it takes exactly t_1 time units to reach the solid saddle path in Fig. 11.4 (which is in fact the same as the saddle path before time t_0). The implied position of the economy at time t_1 is indicated by the point G in the figure.

Immediately after time t_0, $\tilde{k}$ will be decreasing (because saving is smaller than what is required to sustain a constant $\tilde{k}$) and $\tilde{c}$ will be *decreasing* in view of the Keynes-Ramsey rule in a situation with an after-tax rate of return lower than $\rho + \theta g$. Precisely at time t_1, when the temporary tax-transfers scheme based on τ_r' is abolished (as announced and expected), the economy reaches the solid saddle path. From that time the return on saving is high both because of the abolition of the high capital income tax and because $\tilde{k}$ is relatively low. The general equilibrium effect of this is higher saving, and so the economy moves along the solid saddle path back to the original steady-state point E.

There is a last case to consider, namely an anticipated temporary shift in τ_r. We leave that for an exercise, see Exercise 11.??

11.1.4 Ricardian equivalence

We now drop the balanced budget assumption and allow public spending to be financed partly by issuing government bonds and partly by lump-sum taxation. Transfers and gross tax revenue as of time t are called X_t and $\tilde{T}_t$ respectively, while the real value of government net debt is called B_t. For simplicity, we assume all public debt is short-term. Ignoring any money-financing of the spending, the increase per time unit in government debt is identical to the government budget

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deficit:

$$\dot{B}_t = r_t B_t + G_t + X_t - \tilde{T}_t. \tag{11.16}$$

As we ignore uncertainty, on its debt the government has to pay the same interest rate, r_t, as other borrowers.

Along an equilibrium path in the Ramsey model the long-run interest rate necessarily exceeds the long-run GDP growth rate. As we saw in Chapter 6, to remain solvent, the government must then, as a debtor, fulfil a solvency requirement analogous to that of the households in the Ramsey model:

$$\lim_{t\to\infty} B_t e^{-\int_0^t r_s ds} \leq 0. \tag{11.17}$$

This NPG condition says that the debt is in the long run allowed to grow at most at a rate less than the interest rate. As in discrete time, given the accounting relationship (11.16), the NPG condition is equivalent to the intertemporal budget constraint

$$\int_0^\infty (G_t + X_t)e^{-\int_0^t r_s ds} dt \leq \int_0^\infty \tilde{T}_t e^{-\int_0^t r_s ds} dt - B_0. \tag{11.18}$$

This says that the present value of the credibly planned public expenditure cannot exceed government net wealth consisting of the present value of the expected future tax revenues minus initial government debt, i.e., assets minus liabilities.

Assuming that the government does not want to be a net creditor to the private sector in the long run, it will not collect more taxes than is necessary to satisfy (11.18). Hence, we replace "$\leq$" by "$=$" and rearrange to obtain

$$\int_0^\infty \tilde{T}_t e^{-\int_0^t r_s ds} dt = \int_0^\infty (G_t + X_t)e^{-\int_0^t r_s ds} dt + B_0. \tag{11.19}$$

Thus, for a given path of G_t and X_t, the stream of the expected tax revenue must be such that its present value equals the present value of total liabilities on the right-hand-side of (11.19). A temporary budget deficit leads to more debt and therefore also higher taxes in the future. A budget deficit merely implies a deferment of tax payments. The condition (11.19) can be reformulated as

$$\int_0^\infty (\tilde{T}_t - G_t - X_t)e^{-\int_0^t r_s ds} dt = B_0,$$

showing that *if net debt is positive today, then the government has to run a positive primary budget surplus* (that is, $\tilde{T}_t - G_t - X_t > 0$) *in a sufficiently long time in the future.*

We will now show that when taxes are lump sum, then *Ricardian equivalence* holds in the Ramsey model with a public sector.[7] That is, a temporary tax

[7] It is enough that just those taxes that are varied in the thought experiment are lump-sum.

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cut will have no consequences for aggregate consumption. The time profile of lump-sum taxes does not matter.

Consider the intertemporal budget constraint of the representative household,

$$\int_0^\infty c_t L_t e^{-\int_0^t r_s ds} dt \leq A_0 + H_0 = K_0 + B_0 + H_0, \tag{11.20}$$

where H_0 is human wealth of the household. This says, that the present value of the planned consumption stream can not exceed the total wealth of the household. In the optimal plan of the household, we have strict equality in (11.20).

Let τ_t denote the lump-sum per capita *net* tax. Then, $\tilde{T}_t - X_t = \tau_t L_t$ and

$$\begin{aligned} H_0 &= h_0 L_0 = \int_0^\infty (w_t - \tau_t) L_t e^{-\int_0^t r_s ds} dt = \int_0^\infty (w_t L_t + X_t - \tilde{T}_t) e^{-\int_0^t r_s ds} dt \\ &= \int_0^\infty (w_t L_t - G_t) e^{-\int_0^t r_s ds} dt - B_0, \end{aligned} \tag{11.21}$$

where the last equality comes from rearranging (11.19). It follows that

$$B_0 + H_0 = \int_0^\infty (w_t L_t - G_t) e^{-\int_0^t r_s ds} dt.$$

We see that the time profiles of transfers and taxes have fallen out. What matters for total wealth of the forward-looking household is just the spending on goods and services, not the time profile of transfers and taxes. A higher initial debt has no effect on the *sum*, $B_0 + H_0$, because H_0, which incorporates transfers and taxes, becomes equally much lower. Total private wealth is thus unaffected by government debt. So is therefore also private consumption when net taxes are lump sum. A temporary tax cut will not make people feel wealthier and induce them to consume more. Instead they will increase their saving by the same amount as taxes have been reduced, thereby preparing for the higher taxes in the future.

This is the *Ricardian equivalence* result, which we encountered also in Barro's discrete time dynasty model in Chapter 7:

> In a representative agent model with full employment, rational expectations, and no credit market imperfections, if taxes are lump sum, then, for a given evolution of public expenditure, aggregate private consumption is independent of whether current public expenditure is financed by taxes or by issuing bonds. The latter method merely implies a deferment of tax payments. Given the government's intertemporal budget constraint, (11.19), a cut in current taxes has to be offset by a rise in future taxes of the same present value. Since, with lump-sum taxation, it is only the present value of the stream of taxes that matters, the "timing" is irrelevant.

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The assumptions of a representative agent and a long-run interest rate in excess of the long-run GDP growth rate are of key importance. As pointed out in Chapter 6, Ricardian equivalence breaks down in OLG models without an operative Barro-style bequest motive. Such a bequest motive is implicit in the infinite horizon of the Ramsey household. In OLG models, where finite life time is emphasized, there is a turnover in the population of tax payers so that taxes levied at different times are levied on partly different sets of agents. In the future there are newcomers and they will bear part of the higher future tax burden. Therefore, a current tax cut makes current generations feel wealthier and this leads to an increase in current consumption, implying a decrease in national saving, as a result of the temporary deficit finance. The present generations benefit, but future generations bear the cost in the form of smaller national wealth than otherwise. We return to further reasons for absence of Ricardian equivalence in chapters 13 and 19.

11.2 Learning by investing and investment-enhancing policy

In *endogenous growth theory* the Ramsey framework has been applied extensively as a simplifying description of the household sector. In most endogenous growth theory the focus is on mechanisms that generate and shape technological change. Different hypotheses about the generation of new technologies are then often combined with a simplified picture of the household sector as in the Ramsey model. Since this results in a simple determination of the long-run interest rate (the modified golden rule), the analyst can in a first approach concentrate on the main issue, technological change, without being disturbed by aspects that are often secondary to this issue.

As an example, let us consider one of the basic endogenous growth models, the *learning-by-investing model,* sometimes called the *learning-by-doing model.* Learning from investment experience and diffusion across firms of the resulting new technical knowledge (positive externalities) play an important role.

There are two popular alternative versions of the model. The distinguishing feature is whether the learning parameter (see below) is less than one or equal to one. The first case corresponds to a model by Nobel laureate Kenneth Arrow (1962). The second case has been drawn attention to by Paul Romer (1986) who assumes that the learning parameter equals one. We first consider the common framework shared by these two models. Next we describe and analyze Arrow's model (in a simplified version) and finally we compare it to Romer's.

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11.2.1 The common framework

We consider a closed economy with firms and households interacting under conditions of perfect competition. Later, a government attempting to internalize the positive investment externality is introduced.

Let there be N firms in the economy (N "large"). Suppose they all have the same neoclassical production function, F, with CRS. Firm no. i faces the technology

$$Y_{it} = F(K_{it}, \mathcal{T}_t L_{it}), \qquad i = 1, 2, ..., N, \tag{11.22}$$

where the economy-wide technology level $\mathcal{T}_t$ is an increasing function of society's previous experience, proxied by cumulative aggregate net investment:

$$\mathcal{T}_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \quad 0 < \lambda \leq 1, \tag{11.23}$$

where I_s^n is aggregate net investment and $K_t = \sum_i K_{it}$.[8]

The idea is that investment – the production of capital goods – as an unintended *by-product* results in *experience.* The firm and its employees learn from this experience. Producers recognize opportunities for process and quality improvements. In this way knowledge is achieved about how to produce the capital goods in a cost-efficient way and how to design them so that in combination with labor they are more productive and better satisfy the needs of the users. Moreover, as emphasized by Arrow,

> "each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962).[9]

The learning is assumed to benefit essentially all firms in the economy. There are knowledge spillovers across firms and these spillovers are reasonably fast relative to the time horizon relevant for growth theory. In our macroeconomic approach both F and $\mathcal{T}$ are in fact assumed to be exactly the same for all firms in the economy. That is, in this specification the firms producing consumption-goods benefit from the learning just as much as the firms producing capital-goods.

The parameter λ indicates the elasticity of the general technology level, $\mathcal{T}$, with respect to cumulative aggregate net investment and is named the "learning

[8] For arbitrary units of measurement for labor and output the hypothesis is $\mathcal{T}_t = BK_t^{\lambda}$, $B > 0$. In (11.23) measurement units are chosen such that $B = 1$.

[9] Concerning empirical evidence of learning-by-doing and learning-by-investing, see Literature Notes. The citation of Arrow indicates that it was experience from cumulative *gross* investment he had in mind as the basis for learning. Yet, to simplify, we stick to the hypothesis in (11.23), where it is cumulative net investment that matters.

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parameter". Whereas Arrow assumes $\lambda < 1$, Romer focuses on the case $\lambda = 1$. The case of $\lambda > 1$ is ruled out since it would lead to explosive growth (infinite output in finite time) and is therefore not plausible.

The individual firm

In the simple Ramsey model we assumed that households directly own the capital goods in the economy and rent them out to the firms. When discussing learning-by-investment, it somehow fits the intuition better if we (realistically) assume that the firms generally own the capital goods they use. They then finance their capital investment by issuing shares and bonds. Households' financial wealth then consists of these shares and bonds.

Consider firm i. There is perfect competition in all markets. So the firm is a price taker. Its problem is to choose a production and investment plan which maximizes the present value, V_i, of expected future cash-flows. Thus the firm chooses $(L_{it}, I_{it})_{t=0}^{\infty}$ to maximize

$$V_{i0} = \int_0^{\infty} \left[F(K_{it}, \mathcal{T}_t L_{it}) - w_t L_{it} - I_{it}\right] e^{-\int_0^t r_s ds} dt$$

subject to $\dot{K}_{it} = I_{it} - \delta K_{it}$. Here w_t and I_t are the real wage and gross investment, respectively, at time t, r_s is the real interest rate at time s, and $\delta \geq 0$ is the capital depreciation rate. Rising marginal capital installation costs and other kinds of adjustment costs are assumed minor and can be ignored. It can be shown, cf. Chapter 14, that in this case the firm's problem is equivalent to maximization of current pure profits in every short time interval. So, as hitherto, we can describe the firm as just solving a series of static profit maximization problems.

We suppress the time index when not needed for clarity. At any date firm i maximizes current pure profits, $\Pi_i = F(K_i, \mathcal{T} L_i) - (r + \delta)K_i - wL_i$. This leads to the first-order conditions for an interior solution:

$$\begin{aligned} \partial \Pi_i / \partial K_i &= F_1(K_i, \mathcal{T} L_i) - (r + \delta) = 0, \\ \partial \Pi_i / \partial L_i &= F_2(K_i, \mathcal{T} L_i)\mathcal{T} - w = 0. \end{aligned} \tag{11.24}$$

Behind (11.24) is the presumption that each firm is small relative to the economy as a whole, so that each firm's investment has a negligible effect on the economy-wide technology level $\mathcal{T}_t$. Since F is homogeneous of degree one, by Euler's theorem,[10] the first-order partial derivatives, F_1 and F_2, are homogeneous of degree 0. Thus, we can write (11.24) as

$$F_1(k_i, \mathcal{T}) = r + \delta, \tag{11.25}$$

[10] See Math tools.

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where $k_i \equiv K_i/L_i$. Since F is neoclassical, $F_{11} < 0$. Therefore (11.25) determines k_i uniquely. From (11.25) follows that the chosen capital-labor ratio, k_i, will be the same for all firms, say $\bar{k}$.

The individual household

The household sector is described by our standard Ramsey framework with inelastic labor supply and a constant population growth rate $n \geq 0$. The households have CRRA instantaneous utility with parameter $\theta > 0$. The pure rate of time preference is a constant, ρ. The flow budget identity in per capita terms is

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given},$$

where a is per capita financial wealth. The NPG condition is

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0.$$

The resulting consumption-saving plan implies that per capita consumption follows the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho),$$

and the transversality condition that the NPG condition is satisfied with strict equality. In general equilibrium of our closed economy with no role for natural resources and no government debt, a_t will equal K_t/L_t.

Equilibrium in factor markets

For every t we have in equilibrium that $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labor, respectively (both predetermined). Since $K = \sum_i K_i = \sum_i k_i L_i = \sum_i \bar{k} L_i = \bar{k}L$, the chosen capital intensity, k_i, satisfies

$$k_i = \bar{k} = \frac{K}{L} \equiv k, \qquad i = 1, 2, ..., N. \tag{11.26}$$

As a consequence we can use (11.25) to *determine* the equilibrium interest rate:

$$r_t = F_1(k_t, \mathcal{T}_t) - \delta. \tag{11.27}$$

That is, whereas in the firm's first-order condition (11.25) causality goes from r_t to k_{it}, in (11.27) causality goes from k_t to r_t. Note also that in our closed economy with no natural resources and no government debt, a_t will equal k_t.

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The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i F(k_i, \mathcal{T}) L_i = \sum_i F(k, \mathcal{T}) L_i \\ &= F(k, \mathcal{T}) \sum_i L_i = F(k, \mathcal{T}) L = F(K, \mathcal{T}L) = F(K, K^{\lambda} L), \quad (11.28) \end{aligned}$$

where we have used (11.22), (11.26), and (11.23) and the assumption that F is homogeneous of degree one.

11.2.2 The arrow case: $\lambda < 1$

The Arrow case is the robust case where the learning parameter satisfies $0 < \lambda < 1$. The method for analyzing the Arrow case is analogue to that used in the study of the Ramsey model with exogenous technical progress. In particular, aggregate capital per unit of effective labor, $\tilde{k} \equiv K/(\mathcal{T}L)$, is a key variable. Let $\tilde{y} \equiv Y/(\mathcal{T}L)$. Then

$$\tilde{y} = \frac{F(K, \mathcal{T}L)}{\mathcal{T}L} = F(\tilde{k}, 1) \equiv f(\tilde{k}), \qquad f' > 0, f'' < 0. \tag{11.29}$$

We can now write (11.27) as

$$r_t = f'(\tilde{k}_t) - \delta, \tag{11.30}$$

where $\tilde{k}_t$ is pre-determined.

Dynamics

From the definition $\tilde{k} \equiv K/(\mathcal{T}L)$ follows

$$\begin{aligned} \frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{\mathcal{T}}}{\mathcal{T}} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - \lambda \frac{\dot{K}}{K} - n \quad \text{(by (11.23))} \\ &= (1-\lambda)\frac{Y - C - \delta K}{K} - n = (1-\lambda)\frac{\tilde{y} - \tilde{c} - \delta \tilde{k}}{\tilde{k}} - n, \quad \text{where } \tilde{c} \equiv \frac{C}{\mathcal{T}L} \equiv \frac{c}{\mathcal{T}}. \end{aligned}$$

Multiplying through by $\tilde{k}$ we have

$$\dot{\tilde{k}} = (1-\lambda)(f(\tilde{k}) - \tilde{c}) - [(1-\lambda)\delta + n]\,\tilde{k}. \tag{11.31}$$

In view of (11.30), the Keynes-Ramsey rule implies

$$g_c \equiv \frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}\left(f'(\tilde{k}) - \delta - \rho\right). \tag{11.32}$$

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Defining $\tilde{c} \equiv c/A$, now follows

$$\begin{aligned}\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{\mathcal{T}}}{\mathcal{T}} = \frac{\dot{c}}{c} - \lambda\frac{\dot{K}}{K} = \frac{\dot{c}}{c} - \lambda\frac{Y - cL - \delta K}{K} = \frac{\dot{c}}{c} - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}) \\ &= \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}).\end{aligned}$$

Multiplying through by $\tilde{c}$ we have

$$\dot{\tilde{c}} = \left[\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(f(\tilde{k}) - \tilde{c} - \delta\tilde{k})\right]\tilde{c}. \tag{11.33}$$

The two coupled differential equations, (11.31) and (11.33), determine the evolution over time of the economy.

Phase diagram Fig. 11.5 depicts the phase diagram. The $\dot{\tilde{k}} = 0$ locus comes from (11.31), which gives

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}, \tag{11.34}$$

where we realistically may assume that $\delta + n/(1-\lambda) > 0$. As to the $\dot{\tilde{c}} = 0$ locus, we have

$$\begin{aligned}\dot{\tilde{c}} &= 0 \text{ for } \tilde{c} = f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda\theta}(f'(\tilde{k}) - \delta - \rho) \\ &= f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda}g_c \equiv c(\tilde{k}) \quad \text{(from (11.32))}.\end{aligned} \tag{11.35}$$

Before determining the slope of the $\dot{\tilde{c}} = 0$ locus, it is convenient to consider the steady state, $(\tilde{k}^*, \tilde{c}^*)$.

Steady state In a steady state $\tilde{c}$ and $\tilde{k}$ are constant so that the growth rate of C as well as K equals $\dot{A}/A + n$, i.e.,

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{\mathcal{T}}}{\mathcal{T}} + n = \lambda\frac{\dot{K}}{K} + n.$$

Solving gives

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{n}{1-\lambda}.$$

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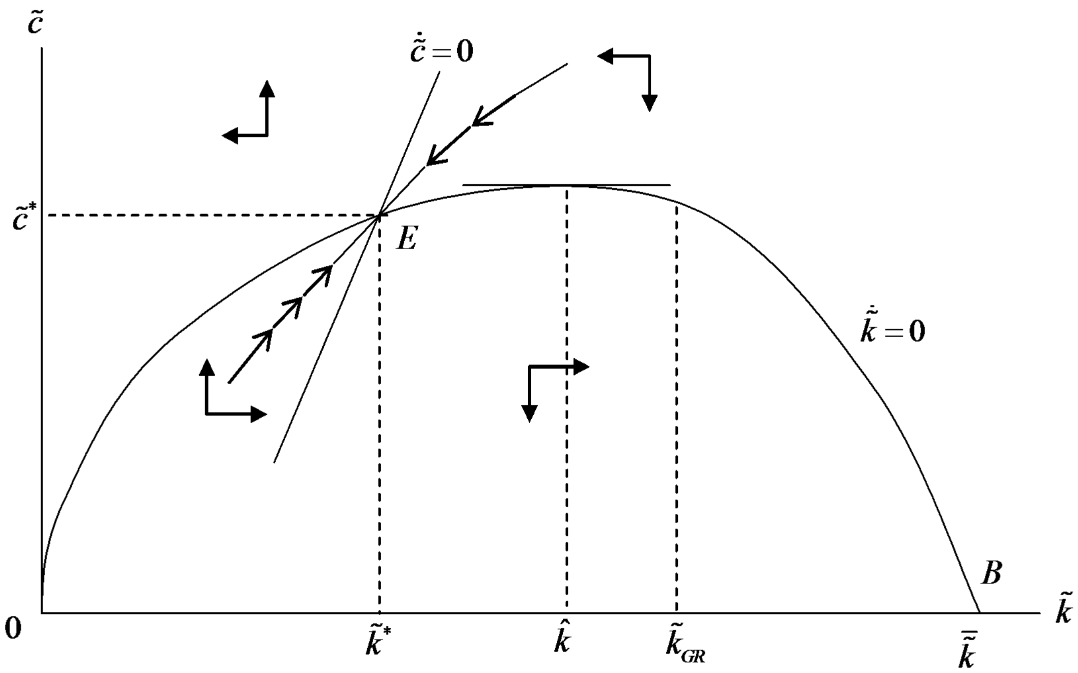

Figure 11.5: Phase diagram for the Arrow model.

Thence, in a steady state

$$g_c = \frac{\dot{C}}{C} - n = \frac{n}{1-\lambda} - n = \frac{\lambda n}{1-\lambda} \equiv g_c^*, \quad \text{and} \tag{11.36}$$

$$\frac{\dot{\mathcal{T}}}{\mathcal{T}} = \lambda\frac{\dot{K}}{K} = \frac{\lambda n}{1-\lambda} = g_c^*. \tag{11.37}$$

The steady-state values of r and $\tilde{k}$, respectively, will therefore satisfy, by (11.32),

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda}. \tag{11.38}$$

To ensure existence of a steady state we assume that the private marginal productivity of capital is sufficiently sensitive to capital per unit of effective labor, from now called the "capital intensity":

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \delta + \rho + \theta\frac{\lambda n}{1-\lambda} > \lim_{\tilde{k}\to\infty} f'(\tilde{k}). \tag{A1}$$

The transversality condition of the representative household is that $\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0$, where a_t is per capita financial wealth. In general equilibrium

$a_t = k_t \equiv \tilde{k}_t \mathcal{T}_t$, where $\mathcal{T}_t$ in steady state grows according to (11.37). Thus, in steady state the transversality condition can be written

$$\lim_{t\to\infty} \tilde{k}^* e^{(g_c^* - r^* + n)t} = 0. \tag{TVC}$$

For this to hold, we need

$$r^* > g_c^* + n = \frac{n}{1-\lambda}, \tag{11.39}$$

by (11.36). In view of (11.38), this is equivalent to

$$\rho - n > (1-\theta)\frac{\lambda n}{1-\lambda}, \tag{A2}$$

which we assume satisfied.

As to the slope of the $\dot{\tilde{c}} = 0$ locus we have from (11.35),

$$c'(\tilde{k}) = f'(\tilde{k}) - \delta - \frac{1}{\lambda}(\tilde{k}\frac{f''(\tilde{k})}{\theta} + g_c) > f'(\tilde{k}) - \delta - \frac{1}{\lambda}g_c, \tag{11.40}$$

since $f'' < 0$. At least in a small neighborhood of the steady state we can sign the right-hand side of this expression. Indeed,

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* = \rho + \theta g_c^* - \frac{1}{\lambda}g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda} - \frac{n}{1-\lambda} = \rho - n - (1-\theta)\frac{\lambda n}{1-\lambda} > 0, \tag{11.41}$$

by (11.36) and (A2). So, combining with (11.40), we conclude that $c'(\tilde{k}^*) > 0$. By continuity, in a small neighborhood of the steady state, $c'(\tilde{k}) \approx c'(\tilde{k}^*) > 0$. Therefore, close to the steady state, the $\dot{\tilde{c}} = 0$ locus is positively sloped, as indicated in Fig. 11.5.

Still, we have to check the following question: In a neighborhood of the steady state, which is steeper, the $\dot{\tilde{c}} = 0$ locus or the $\dot{\tilde{k}} = 0$ locus? The slope of the latter is $f'(\tilde{k}) - \delta - n/(1-\lambda)$, from (11.34). At the steady state this slope is

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* \in (0, c'(\tilde{k}^*)),$$

in view of (11.41) and (11.40). The $\dot{\tilde{c}} = 0$ locus is thus steeper. So, the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus from below and can only cross once.

The assumption (A1) ensures existence of a $\tilde{k}^* > 0$ satisfying (11.38). As Fig. 11.5 is drawn, a little more is implicitly assumed namely that there exists a

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$\hat{k} > 0$ such that the *private* net marginal productivity of capital equals the the steady-state growth rate of output, i.e.,

$$f'(\hat{k}) - \delta = (\frac{\dot{Y}}{Y})^* = (\frac{\dot{\mathcal{T}}}{\mathcal{T}})^* + \frac{\dot{L}}{L} = \frac{\lambda n}{1-\lambda} + n = \frac{n}{1-\lambda}, \tag{11.42}$$

where we have used (11.37). Thus, the tangent to the $\dot{\tilde{k}} = 0$ locus at $\tilde{k} = \hat{k}$ is horizontal and $\hat{k} > \tilde{k}^*$ as indicated in the figure.

Note, however, that $\hat{k}$ is not the golden-rule capital intensity. The latter is the capital intensity, $\tilde{k}_{GR}$, at which the *social* net marginal productivity of capital equals the steady-state growth rate of output (see Appendix). If $\tilde{k}_{GR}$ exists, it will be larger than $\hat{k}$ as indicated in Fig. 11.5. To see this, we now derive a convenient expression for the social marginal productivity of capital. From (11.28) we have

$$\begin{aligned}
\frac{\partial Y}{\partial K} &= F_1(\cdot) + F_2(\cdot)\lambda K^{\lambda-1}L = f'(\tilde{k}) + F_2(\cdot)K^{\lambda}L(\lambda K^{-1}) \quad \text{(by (11.29))} \\
&= f'(\tilde{k}) + (F(\cdot) - F_1(\cdot)K)\,\lambda K^{-1} \qquad \text{(by Euler's theorem)} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda}L - f'(\tilde{k})K)\lambda K^{-1} \qquad \text{(by (11.29) and (11.23))} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda-1}L - f'(\tilde{k}))\lambda = f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}} > f'(\tilde{k}).
\end{aligned}$$

in view of $\tilde{k} = K/(K^{\lambda}L) = K^{1-\lambda}L^{-1}$ and $f(\tilde{k})/\tilde{k} - f'(\tilde{k}) > 0$. As expected, the positive externality makes the social marginal productivity of capital larger than the private one. Since we can also write $\partial Y/\partial K = (1-\lambda)f'(\tilde{k}) + \lambda f(\tilde{k})/\tilde{k}$, we see that $\partial Y/\partial K$ is a decreasing function of $\tilde{k}$ (both $f'(\tilde{k})$ and $f(\tilde{k})/\tilde{k}$ are decreasing in $\tilde{k}$.

Now, the golden-rule capital intensity, $\tilde{k}_{GR}$, will be that capital intensity which satisfies

$$f'(\tilde{k}_{GR}) + \lambda\frac{f(\tilde{k}_{GR}) - \tilde{k}_{GR}f'(\tilde{k}_{GR})}{\tilde{k}_{GR}} - \delta = (\frac{\dot{Y}}{Y})^* = \frac{n}{1-\lambda}.$$

To ensure there exists such a $\tilde{k}_{GR}$, we strengthen the right-hand side inequality in (A1) by the assumption

$$\lim_{\tilde{k}\to\infty}\left(f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}}\right) < \delta + \frac{n}{1-\lambda}. \tag{A3}$$

This, together with (A1) and $f'' < 0$, implies existence of a unique $\tilde{k}_{GR}$, and in view of our additional assumption (A2), we have $0 < \tilde{k}^* < \hat{k} < \tilde{k}_{GR}$, as displayed in Fig. 11.5.

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Stability The arrows in Fig. 11.5 indicate the direction of movement as determined by (11.31) and (11.33). We see that the steady state is a saddle point. The dynamic system has one pre-determined variable, $\tilde{k}$, and one jump variable, $\tilde{c}$. The saddle path is not parallel to the jump variable axis. We claim that for a given $\tilde{k}_0 > 0$, (i) the initial value of $\tilde{c}_0$ will be the ordinate to the point where the vertical line $\tilde{k} = \tilde{k}_0$ crosses the saddle path; (ii) over time the economy will move along the saddle path towards the steady state. Indeed, this time path is consistent with all conditions of general equilibrium, including the transversality condition (TVC). And the path is the *only* technically feasible path with this property. Indeed, all the divergent paths in Fig. 11.5 can be ruled out as equilibrium paths because they can be shown to violate the transversality condition of the household.

In the long run c and $y \equiv Y/L \equiv \tilde{y}\mathcal{T} = f(\tilde{k}^*)\mathcal{T}$ grow at the rate $\lambda n/(1-\lambda)$, which is positive if and only if $n > 0$. This is an example of *endogenous growth* in the sense that the positive long-run per capita growth rate is generated through an internal mechanism (learning) in the model (in contrast to exogenous technology growth as in the Ramsey model with exogenous technical progress).

Two types of endogenous growth

One may distinguish between two types of endogenous growth. One is called *fully endogenous* growth which occurs when the long-run growth rate of c is positive without the support by growth in any exogenous factor (for example exogenous growth in the labor force); the Romer case, to be considered in the next section, provides an example. The other type is called *semi-endogenous growth* and is present if growth is endogenous but a positive per capita growth rate can not be maintained in the long run without the support by growth in some exogenous factor (for example growth in the labor force). Clearly, in the Arrow model of learning by investing, growth is "only" semi-endogenous. The technical reason for this is the assumption that the learning parameter λ is below 1, which implies diminishing returns to capital at the aggregate level. If and only if $n > 0$, do we have $\dot{c}/c > 0$ in the long run.[11] In line with this, $\partial g_y^*/\partial n > 0$.

The key role of population growth derives from the fact that although there are diminishing marginal returns to capital at the aggregate level, there are increasing returns to scale w.r.t. capital *and* labor. For the increasing returns to be exploited, growth in the labor force is needed. To put it differently: when there are increasing returns to K and L together, growth in the labor force not only counterbalances the falling marginal productivity of aggregate capital (this

[11] Note, however, that the model, and therefore (11.36), presupposes $n \geq 0$. If $n < 0$, then K would tend to be decreasing and so, by (11.23), the level of technical knowledge would be decreasing, which is implausible, at least for a modern industrialized economy.

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counter-balancing role reflects the complementarity between K and L), but also upholds sustained productivity growth.

Note that in the semi-endogenous growth case $\partial g_y^*/\partial\lambda = n/(1-\lambda)^2 > 0$ for $n > 0$. That is, a higher value of the learning parameter implies higher per capita growth in the long run, when $n > 0$. Note also that $\partial g_y^*/\partial\rho = 0 = \partial g_y^*/\partial\theta$, that is, in the semi-endogenous growth case preference parameters do not matter for long-run growth. As indicated by (11.36), the long-run growth rate is tied down by the learning parameter, λ, and the rate of population growth, n. But, like in the simple Ramsey model, it can be shown that preference parameters matter for the *level* of the growth path. This suggests that taxes and subsidies do not have long-run growth effects, but "only" *level* effects (see Exercise 11.??).

11.2.3 Romer's limiting case: $\lambda = 1$, $n = 0$

We now consider the limiting case $\lambda = 1$. We should think of it as a thought experiment because, by most observers, the value 1 is considered an unrealistically high value for the learning parameter. To avoid a forever rising growth rate we have to add the restriction $n = 0$.

The resulting model turns out to be extremely simple and at the same time it gives striking results (both circumstances have probably contributed to its popularity).

First, with $\lambda = 1$ we get $\mathcal{T} = K$ and so the equilibrium interest rate is, by (11.27),

$$r = F_1(k, K) - \delta = F_1(1, L) - \delta \equiv \bar{r},$$

where we have divided the two arguments of $F_1(k, K)$ by $k \equiv K/L$ and again used Euler's theorem. Note that the interest rate is constant "from the beginning" and independent of the historically given initial value of K, K_0. The aggregate production function is now

$$Y = F(K, KL) = F(1, L)K, \qquad L \text{ constant}, \tag{11.43}$$

and is thus *linear* in the aggregate capital stock. In this way the general neoclassical presumption of diminishing returns to capital has been suspended and replaced by exactly constant returns to capital. So the Romer model belongs to a class of models known as *AK models*, that is, models where in general equilibrium the interest rate and the output-capital ratio are necessarily constant over time whatever the initial conditions.

The method for analyzing an AK model is different from the one used for a diminishing returns model as above.

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Dynamics

The Keynes-Ramsey rule now takes the form

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\bar{r} - \rho) = \frac{1}{\theta}(F_1(1, L) - \delta - \rho) \equiv \gamma, \tag{11.44}$$

which is also constant "from the beginning". To ensure positive growth, we assume

$$F_1(1, L) - \delta > \rho. \tag{A1'}$$

And to ensure bounded intertemporal utility (and existence of equilibrium), it is assumed that

$$\rho > (1 - \theta)\gamma \text{ and therefore } \gamma < \theta\gamma + \rho = \bar{r}. \tag{A2'}$$

Solving the linear differential equation (11.44) gives

$$c_t = c_0 e^{\gamma t}, \tag{11.45}$$

where c_0 is unknown so far (because c is not a predetermined variable). We shall find c_0 by applying the households' transversality condition

$$\lim_{t\to\infty} a_t e^{-\bar{r}t} = \lim_{t\to\infty} k_t e^{-\bar{r}t} = 0. \tag{TVC}$$

First, note that the dynamic resource constraint for the economy is

$$\dot{K} = Y - cL - \delta K = F(1, L)K - cL - \delta K,$$

or, in per-capita terms,

$$\dot{k} = [F(1, L) - \delta]\, k - c_0 e^{\gamma t}. \tag{11.46}$$

In this equation it is important that $F(1, L) - \delta - \gamma > 0$. To understand this inequality, note that, by (A2'), $F(1, L) - \delta - \gamma > F(1, L) - \delta - \bar{r} = F(1, L) - F_1(1, L)$ $= F_2(1, L)L > 0$, where the first equality is due to $\bar{r} = F_1(1, L) - \delta$ and the second is due to the fact that since F is homogeneous of degree 1, we have, by Euler's theorem, $F(1, L) = F_1(1, L) \cdot 1 + F_2(1, L)L > F_1(1, L) > \delta$, in view of (A1'). The key property $F(1, L) - F_1(1, L) > 0$ is illustrated in Fig. 11.6.

The solution of a linear differential equation of the form $\dot{x}(t) + ax(t) = ce^{ht}$, with $h \neq -a$, is

$$x(t) = (x(0) - \frac{c}{a + h})e^{-at} + \frac{c}{a + h}e^{ht}. \tag{11.47}$$

Thus the solution to (11.46) is

$$k_t = (k_0 - \frac{c_0}{F(1, L) - \delta - \gamma})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1, L) - \delta - \gamma}e^{\gamma t}. \tag{11.48}$$

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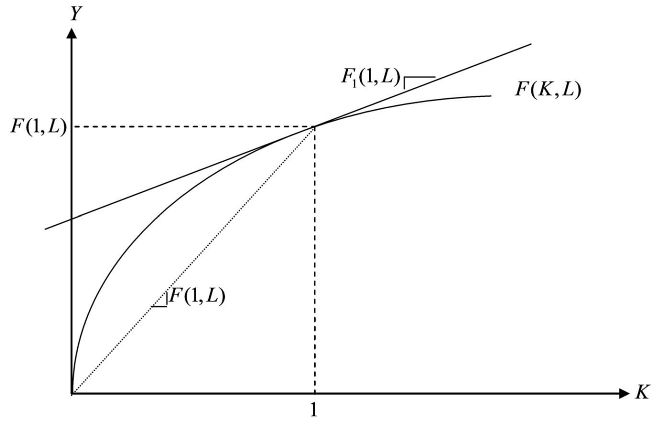

Figure 11.6: Illustration of the fact that for L given, $F(1,L) > F_1(1,L)$.

To check whether (TVC) is satisfied we consider

$$\begin{aligned} k_t e^{-\bar{r}t} &= (k_0 - \frac{c_0}{F(1,L)-\delta-\gamma})e^{(F(1,L)-\delta-\bar{r})t} + \frac{c_0}{F(1,L)-\delta-\gamma}e^{(\gamma-\bar{r})t} \\ &\to (k_0 - \frac{c_0}{F(1,L)-\delta-\gamma})e^{(F(1,L)-\delta-\bar{r})t} \text{ for } t \to \infty, \end{aligned}$$

since $\bar{r} > \gamma$, by (A2'). But $\bar{r} = F_1(1,L) - \delta < F(1,L) - \delta$, and so (TVC) is only satisfied if

$$c_0 = (F(1,L) - \delta - \gamma)k_0. \tag{11.49}$$

If c_0 is less than this, there will be over-saving and (TVC) is violated ($a_t e^{-\bar{r}t} \to \infty$ for $t \to \infty$, since $a_t = k_t$). If c_0 is higher than this, both the NPG and (TVC) are violated ($a_t e^{-\bar{r}t} \to -\infty$ for $t \to \infty$).

Inserting the solution for c_0 into (11.48), we get

$$k_t = \frac{c_0}{F(1,L)-\delta-\gamma}e^{\gamma t} = k_0 e^{\gamma t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1,L)k$, the same is true for y. Hence, from start the system is in balanced growth (there is no transitional dynamics).

This is a case of *fully endogenous growth* in the sense that the long-run growth rate of c is positive without the support by growth in any exogenous factor. This outcome is due to the absence of diminishing returns to aggregate capital, which is implied by the assumed high value of the learning parameter. The empirical foundation for being in a neighborhood of this high value is weak, however, cf. Literature notes. A further problem with this special version of the learning model is that the results are *non-robust*. With λ slightly less than 1, we are back

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in the Arrow case and growth peters out, since $n = 0$. With λ slightly above 1, it can be shown that growth becomes explosive (infinite output in finite time).[12]

The Romer case, $\lambda = 1$, is thus a *knife-edge* case in a double sense. First, it imposes a particular value for a parameter which *apriori* can take any value within an interval. Second, the imposed value leads to theoretically non-robust results; values in a hair's breadth distance result in qualitatively different behavior of the dynamic system. Still, whether the Romer case - or, more generally, a fully-endogenous growth case - can be used as an empirical approximation to its semi-endogenous "counterpart" for a sufficiently long time horizon to be of interest, is a debated question within growth analysis.

It is noteworthy that the *causal structure* in the long run in the diminishing returns case is different than in the AK-case of Romer. In the diminishing returns case the steady-state growth rate is determined first, as g_c^* in (11.36), and then r^* is determined through the Keynes-Ramsey rule; finally, Y/K is determined by the technology, given r^*. In contrast, the Romer case has Y/K and r directly given as $F(1, L)$ and $\bar{r}$, respectively. In turn, $\bar{r}$ determines the (constant) equilibrium growth rate through the Keynes-Ramsey rule.

Economic policy in the Romer case

In the AK case, that is, the fully endogenous growth case, we have $\partial\gamma/\partial\rho < 0$ and $\partial\gamma/\partial\theta < 0$. Thus, preference parameters *matter* for the long-run growth rate and not "only" for the *level* of the growth path. This suggests that taxes and subsidies can have *long-run* growth effects. In any case, in this model there is a motivation for government intervention due to the positive externality of private investment. This motivation is present whether $\lambda < 1$ or $\lambda = 1$. Here we concentrate on the latter case, which is the simpler one. We first find the social planner's solution.

The social planner The social planner faces the aggregate production function $Y_t = F(1, L)K_t$ or, in per capita terms, $y_t = F(1, L)k_t$. The social planner's problem is to choose $(c_t)_{=0}^{\infty}$ to maximize

$$
\begin{aligned}
U_0 &= \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\
c_t &\geq 0, \\
\dot{k}_t &= F(1, L)k_t - c_t - \delta k_t, \qquad k_0 > 0 \text{ given}, && (11.50) \\
k_t &\geq 0 \text{ for all } t > 0. && (11.51)
\end{aligned}
$$

[12] See Solow (1997).

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The current-value Hamiltonian is

$$H(k, c, \eta, t) = \frac{c^{1-\theta}}{1-\theta} + \eta\left(F(1, L)k - c - \delta k\right),$$

where $\eta = \eta_t$ is the adjoint variable associated with the state variable, which is capital per unit of labor. Necessary first-order conditions for an interior optimal solution are

$$\frac{\partial H}{\partial c} = c^{-\theta} - \eta = 0, \text{ i.e., } c^{-\theta} = \eta, \tag{11.52}$$

$$\frac{\partial H}{\partial k} = \eta(F(1, L) - \delta) = -\dot{\eta} + \rho\eta. \tag{11.53}$$

We guess that also the transversality condition,

$$\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0, \tag{11.54}$$

must be satisfied by an optimal solution. This guess will be of help in finding a candidate solution. Having found a candidate solution, we shall invoke a theorem on *sufficient* conditions to ensure that our candidate solution *is* really a solution.

Log-differentiating w.r.t. t in (11.52) and combining with (11.53) gives the social planner's Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(F(1, L) - \delta - \rho) \equiv \gamma_{SP}. \tag{11.55}$$

We see that $\gamma_{SP} > \gamma$. This is because the social planner internalizes the economy-wide learning effect associated with capital investment, that is, the social planner takes into account that the "social" marginal productivity of capital is $\partial y_t/\partial k_t = F(1, L) > F_1(1, L)$. To ensure bounded intertemporal utility we sharpen (A2') to

$$\rho > (1 - \theta)\gamma_{SP}. \tag{A2"}$$

To find the time path of k_t, note that the dynamic resource constraint (11.50) can be written

$$\dot{k}_t = (F(1, L) - \delta)k_t - c_0 e^{\gamma_{SP} t},$$

in view of (11.55). By the general solution formula (11.47) this has the solution

$$k_t = (k_0 - \frac{c_0}{F(1, L) - \delta - \gamma_{SP}})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} e^{\gamma_{SP} t}. \tag{11.56}$$

In view of (11.53), in an interior optimal solution the time path of the adjoint variable η is

$$\eta_t = \eta_0 e^{-[(F(1,L)-\delta-\rho]t},$$

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where $\eta_0 = c_0^{-\theta} > 0$, by (11.52). Thus, the conjectured transversality condition (11.54) implies

$$\lim_{t\to\infty} k_t e^{-(F(1,L)-\delta)t} = 0, \tag{11.57}$$

where we have eliminated η_0. To ensure that this is satisfied, we multiply k_t from (11.56) by $e^{-(F(1,L)-\delta)t}$ to get

$$\begin{aligned} k_t e^{-(F(1,L)-\delta)t} &= k_0 - \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} + \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} e^{[\gamma_{SP}-(F(1,L)-\delta)]t} \\ &\to k_0 - \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} \text{ for } t \to \infty, \end{aligned}$$

since, by (A2"), $\gamma_{SP} < \rho + \theta\gamma_{SP} = F(1,L) - \delta$ in view of (11.55). Thus, (11.57) is only satisfied if

$$c_0 = (F(1,L) - \delta - \gamma_{SP})k_0. \tag{11.58}$$

Inserting this solution for c_0 into (11.56), we get

$$k_t = \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} e^{\gamma_{SP}t} = k_0 e^{\gamma_{SP}t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1,L)k$, the same is true for y. Hence, our candidate for the social planner's solution is from start in balanced growth (there is no transitional dynamics).

The next step is to check whether our candidate solution satisfies a set of *sufficient* conditions for an optimal solution. Here we can use *Mangasarian's theorem*. Applied to a continuous-time optimization problem like this, with one control variable and one state variable, the theorem says that the following conditions are sufficient:

(a) Concavity: For all $t \geq 0$ the Hamiltonian is jointly concave in the control and state variables, here c and k.

(b) Non-negativity: There is for all $t \geq 0$ a non-negativity constraint on the state variable; in addition, the co-state variable, η, is non-negative for all $t \geq 0$ along the optimal path.

(c) TVC: The candidate solution satisfies the transversality condition $\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0$, where $\eta_t e^{-\rho t}$ is the discounted co-state variable.

In the present case we see that the Hamiltonian is a sum of concave functions and therefore is itself concave in (k, c). Further, from (11.51) we see that condition (b) is satisfied. Finally, our candidate solution is constructed so as to satisfy condition (c). The conclusion is that our candidate solution *is* an optimal solution. We call it an *SP allocation.*

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Implementing the SP allocation in the market economy Returning to the competitive market economy, we assume there is a policy maker, the government, with only two activities. These are (i) paying an investment subsidy, s, to the firms so that their capital costs are reduced to

$$(1-s)(r+\delta)$$

per unit of capital per time unit; (ii) financing this subsidy by a constant consumption tax rate τ.

Let us first find the size of s needed to establish the SP allocation. Firm i now chooses K_i such that

$$\frac{\partial Y_i}{\partial K_i}|_{K \text{ fixed}} = F_1(K_i, KL_i) = (1-s)(r+\delta).$$

By Euler's theorem this implies

$$F_1(k_i, K) = (1-s)(r+\delta) \quad \text{for all } i,$$

so that in equilibrium we must have

$$F_1(k, K) = (1-s)(r+\delta),$$

where $k \equiv K/L$, which is pre-determined from the supply side. Thus, the equilibrium interest rate must satisfy

$$r = \frac{F_1(k, K)}{1-s} - \delta = \frac{F_1(1, L)}{1-s} - \delta, \tag{11.59}$$

again using Euler's theorem.

It follows that s should be chosen such that the "right" r arises. What is the "right" r? It is that net rate of return which is implied by the production technology at the aggregate level, namely $\partial Y/\partial K - \delta = F(1, L) - \delta$. If we can obtain $r = F(1, L) - \delta$, then there is no wedge between the intertemporal rate of transformation faced by the consumer and that implied by the technology. The required s thus satisfies

$$r = \frac{F_1(1, L)}{1-s} - \delta = F(1, L) - \delta,$$

so that

$$s = 1 - \frac{F_1(1, L)}{F(1, L)} = \frac{F(1, L) - F_1(1, L)}{F(1, L)} = \frac{F_2(1, L)L}{F(1, L)}.$$

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It remains to find the required consumption tax rate τ. The tax revenue will be τcL, and the *required* tax revenue is

$$\mathcal{T} = s(r+\delta)K = \left(F(1,L) - F_1(1,L)\right)K = \tau cL.$$

Thus, with a balanced budget the required tax rate is

$$\tau = \frac{\mathcal{T}}{cL} = \frac{F(1,L) - F_1(1,L)}{c/k} = \frac{F(1,L) - F_1(1,L)}{F(1,L) - \delta - \gamma_{SP}} > 0, \tag{11.60}$$

where we have used that the proportionality in (11.58) between c and k holds for all $t \geq 0$. Substituting (11.55) into (11.60), the solution for τ can be written

$$\tau = \frac{\theta\left[F(1,L) - F_1(1,L)\right]}{(\theta-1)(F(1,L)-\delta)+\rho} = \frac{\theta F_2(1,L)L}{(\theta-1)(F(1,L)-\delta)+\rho}.$$

The required tax rate on consumption is thus a constant. It therefore does not distort the consumption/saving decision on the margin, cf. Appendix B.

It follows that the allocation obtained by this subsidy-tax policy *is* the SP allocation. A policy, here the policy (s, τ), which in a decentralized system induces the SP allocation, is called a *first-best policy.* In a situation where for some reason it is impossible to obtain an SP allocation in a decentralized way (because of adverse selection and moral hazard problems, say), a government's optimization problem would involve additional constraints to those given by technology and initial resources. A decentralized implementation of the solution to such a problem is called a *second-best policy.*

11.3 Concluding remarks

(not yet available)

11.4 Literature notes

(incomplete)

As to empirical evidence of learning-by-doing and learning-by-investing, see ...

As noted in Section 11.2.1, the citation of Arrow indicates that it was experience from cumulative *gross* investment, rather than net investment, he had in mind as the basis for learning. Yet the hypothesis in (11.23) is the more popular one - seemingly for no better reason than that it leads to simpler dynamics.

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Another way in which (11.23) deviates from Arrow's original ideas is by assuming that technical progress is disembodied rather than embodied, a distinction we touched upon in Chapter 2. Moreover, we have assumed a neoclassical technology whereas Arrow assumed fixed technical coefficients.

11.5 Appendix

A. The golden-rule capital intensity in Arrow's growth model

In our discussion of Arrow's learning-by-investing model in Section 11.2.2 (where $0 < \lambda < 1$), we claimed that the golden-rule capital intensity, $\tilde{k}_{GR}$, will be that effective capital-labor ratio at which the social net marginal productivity of capital equals the steady-state growth rate of output. In this respect the Arrow model with endogenous technical progress is similar to the standard neoclassical growth model with exogenous technical progress. This claim corresponds to a very general theorem, valid also for models with many capital goods and non-existence of an aggregate production function. This theorem says that the highest sustainable path for consumption per unit of labor in the economy will be that path which results from those techniques which profit maximizing firms choose under perfect competition when the real interest rate equals the steady-state growth rate of GNP (see Gale and Rockwell, 1975).

To prove our claim, note that in steady state, (11.35) holds whereby consumption per unit of labor (here the same as per capita consumption as L = labor force = population) can be written

$$\begin{aligned}
c_t &\equiv \tilde{c}_t T_t = \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] K_t^{\lambda} \\
&= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \left(K_0 e^{\frac{n}{1-\lambda}t} \right)^{\lambda} \quad (\text{by } g_K^* = \frac{n}{1-\lambda}) \\
&= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \left((\tilde{k}L_0)^{\frac{1}{1-\lambda}} e^{\frac{n}{1-\lambda}t} \right)^{\lambda} \quad (\text{from } \tilde{k} = \frac{K_t}{K_t^{\lambda} L_t} = \frac{K_0^{1-\lambda}}{L_0}) \\
&= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \tilde{k}^{\frac{\lambda}{1-\lambda}} L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t} \equiv \varphi(\tilde{k}) L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t},
\end{aligned}$$

defining $\varphi(\tilde{k})$ in the obvious way.

We look for that value of $\tilde{k}$ at which this steady-state path for c_t is at the highest technically feasible level. The positive coefficient, $L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t}$, is the only time dependent factor and can be ignored since it is exogenous. The problem is thereby reduced to the static problem of maximizing $\varphi(\tilde{k})$ with respect to $\tilde{k} > 0$.

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We find

$$\begin{aligned}\varphi'(\tilde{k}) &= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\right]\tilde{k}^{\frac{\lambda}{1-\lambda}} + \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right]\frac{\lambda}{1-\lambda}\tilde{k}^{\frac{\lambda}{1-\lambda}-1} \\ &= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda}) + \left(\frac{f(\tilde{k})}{\tilde{k}} - (\delta + \frac{n}{1-\lambda})\right)\frac{\lambda}{1-\lambda}\right]\tilde{k}^{\frac{\lambda}{1-\lambda}} \\ &= \left[(1-\lambda)f'(\tilde{k}) - (1-\lambda)\delta - n + \lambda\frac{f(\tilde{k})}{\tilde{k}} - \lambda(\delta + \frac{n}{1-\lambda})\right]\frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \\ &= \left[(1-\lambda)f'(\tilde{k}) - \delta + \lambda\frac{f(\tilde{k})}{\tilde{k}} - \frac{n}{1-\lambda}\right]\frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \equiv \psi(\tilde{k})\frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda}, \qquad (11.61)\end{aligned}$$

defining $\psi(\tilde{k})$ in the obvious way. The first-order condition for the problem, $\varphi'(\tilde{k}) = 0$, is equivalent to $\psi(\tilde{k}) = 0$. After ordering this gives

$$f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}} - \delta = \frac{n}{1-\lambda}. \qquad (11.62)$$

We see that

$$\varphi'(\tilde{k}) \gtreqless 0 \quad \text{for} \quad \psi(\tilde{k}) \gtreqless 0,$$

respectively. Moreover,

$$\psi'(\tilde{k}) = (1-\lambda)f''(\tilde{k}) - \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}^2} < 0,$$

in view of $f'' < 0$ and $f(\tilde{k})/\tilde{k} > f'(\tilde{k})$. So a $\tilde{k} > 0$ satisfying $\psi(\tilde{k}) = 0$ is the unique maximizer of $\varphi(\tilde{k})$. By (A1) and (A3) in Section 11.2.2 such a $\tilde{k}$ exists and is thereby the same as the $\tilde{k}_{GR}$ we were looking for.

The left-hand side of (11.62) equals the social marginal productivity of capital and the right-hand side equals the steady-state growth rate of output. At $\tilde{k} = \tilde{k}_{GR}$ it therefore holds that

$$\frac{\partial Y}{\partial K} - \delta = \left(\frac{\dot{Y}}{Y}\right)^*.$$

This confirms our claim in Section 11.2.2 about $\tilde{k}_{GR}$.

Remark about the absence of a golden rule in the Romer case. In the Romer case the golden rule is not a well-defined concept for the following reason. Along any balanced growth path we have from (11.50),

$$g_k \equiv \frac{\dot{k}_t}{k_t} = F(1, L) - \delta - \frac{c_t}{k_t} = F(1, L) - \delta - \frac{c_0}{k_0},$$

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because g_k $(= g_K)$ is by definition constant along a balanced growth path, whereby also c_t/k_t must be constant. We see that g_k is decreasing linearly from $F(1, L) - \delta$ to $-\delta$ when c_0/k_0 rises from nil to $F(1, L)$. So choosing among alternative technically feasible balanced growth paths is inevitably a choice between starting with low consumption to get high growth forever or starting with high consumption to get low growth forever. Given any $k_0 > 0$, the alternative possible balanced growth paths will therefore sooner or later cross each other in the $(t, \ln c)$ plane. Hence, for the given k_0, there exists no balanced growth path which for all $t \geq 0$ has c_t higher than along any other technically feasible balanced growth path.

B. Consumption taxation

Is a consumption tax distortionary - always? never? sometimes?

The answer is the following.

1. Suppose labor supply is *elastic* (due to leisure entering the utility function). Then a consumption tax (whether constant or time-dependent) is generally distortionary (not neutral). This is because it reduces the effective opportunity cost of leisure by reducing the amount of consumption forgone by working one hour less. Indeed, the tax makes consumption goods more expensive and so the amount of consumption that the agent can buy for the hourly wage becomes smaller. The substitution effect on leisure of a consumption tax is thus positive, while the income and wealth effects will be negative. Generally, the net effect will not be zero, but can be of any sign; it may be small in absolute terms.

2. Suppose labor supply is *inelastic* (no trade-off between consumption and leisure). Then, at least in the type of growth models we consider in this course, a constant (time-independent) consumption tax acts as a lump-sum tax and is thus non-distortionary. If the consumption tax is *time-dependent*, however, a distortion of the *intertemporal* aspect of household decisions tends to arise.

To understand answer 2, consider a Ramsey household with inelastic labor supply. Suppose the household faces a time-varying consumption tax rate $\tau_t > 0$. To obtain a consumption level per time unit equal to c_t per capita, the household has to spend

$$\bar{c}_t = (1 + \tau_t)c_t$$

units of account (in real terms) per capita. Thus, spending $\bar{c}_t$ per capita per time unit results in the per capita consumption level

$$c_t = (1 + \tau_t)^{-1}\bar{c}_t. \tag{11.63}$$

In order to concentrate on the consumption tax as such, we assume the tax revenue is simply given back as lump-sum transfers and that there are no other government activities. Then, with a balanced government budget, we have

$$x_t L_t = \tau_t c_t L_t,$$

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where x_t is the per capita lump-sum transfer, exogenous to the household, and L_t is the size of the representative household.

Assuming CRRA utility with parameter $\theta > 0$, the instantaneous per capita utility can be written

$$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta} = \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta}}{1-\theta}.$$

In our standard notation the household's intertemporal optimization problem is then to choose $(\bar{c}_t)_{t=0}^{\infty}$ so as to maximize

$$\begin{aligned} U_0 &= \int_0^\infty \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} \\ \bar{c}_t &\geq 0, \\ \dot{a}_t &= (r_t - n)a_t + w_t + x_t - \bar{c}_t, \quad a_0 \text{ given}, \\ \lim_{t\to\infty} a_t e^{-\int_0^\infty (r_s-n)ds} &\geq 0. \end{aligned}$$

From now, we let the timing of the variables be implicit unless needed for clarity. The current-value Hamiltonian is

$$H = \frac{(1+\tau)^{\theta-1}\bar{c}^{1-\theta}}{1-\theta} + \lambda\left[(r-n)a + w + x - \bar{c}\right],$$

where λ is the co-state variable associated with financial per capita wealth, a. An interior optimal solution will satisfy the first-order conditions

$$\frac{\partial H}{\partial \bar{c}} = (1+\tau)^{\theta-1}\bar{c}^{-\theta} - \lambda = 0, \text{ so that } (1+\tau)^{\theta-1}\bar{c}^{-\theta} = \lambda, \quad \text{(FOC1)}$$

$$\frac{\partial H}{\partial a} = \lambda(r-n) = -\dot{\lambda} + (\rho - n)\lambda, \quad \text{(FOC2)}$$

and a transversality condition which amounts to

$$\lim_{t\to\infty} a_t e^{-\int_0^\infty (r_s-n)ds} = 0. \quad \text{(TVC)}$$

We take logs in (FOC1) to get

$$(\theta - 1)\log(1+\tau) - \theta \log \bar{c} = \log \lambda.$$

Differentiating w.r.t. time, taking into account that $\tau = \tau_t$, gives

$$(\theta-1)\frac{\dot{\tau}}{1+\tau} - \theta\frac{\dot{\bar{c}}}{\bar{c}} = \frac{\dot{\lambda}}{\lambda} = \rho - r.$$

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By ordering, we find the growth rate of consumption spending,

$$\frac{\dot{\bar{c}}}{\bar{c}} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right].$$

Using (11.63), this gives the growth rate of consumption,

$$\frac{\dot{c}}{c} = \frac{\dot{\bar{c}}}{\bar{c}} - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right] - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}(r - \frac{\dot{\tau}}{1+\tau} - \rho).$$

Assuming firms maximize profit under perfect competition, in equilibrium the real interest rate will satisfy

$$r = \frac{\partial Y}{\partial K} - \delta. \tag{11.64}$$

But the *effective* real interest rate, $\hat{r}$, faced by the consuming household, is

$$\hat{r} = r - \frac{\dot{\tau}}{1+\tau} \lesseqgtr r \text{ for } \dot{\tau} \gtreqless 0,$$

respectively. If for example the consumption tax is increasing, then the effective real interest rate faced by the consumer is smaller than the market real interest rate, given in (11.64), because saving implies postponing consumption and future consumption is more expensive due to the higher consumption tax rate.

The conclusion is that a time-varying consumption tax rate is distortionary. It implies a wedge between the intertemporal rate of transformation faced by the consumer, reflected by $\hat{r}$, and the intertemporal rate of transformation offered by the technology of society, indicated by r in (11.64). On the other hand, *if* the consumption tax rate is constant, the consumption tax is non-distortionary when there is no utility from leisure.

A remark on tax smoothing

Outside steady state it is often so that maintaining constant tax rates is inconsistent with maintaining a balanced government budget. Is the implication of this that we should recommend the government to let tax rates be continually adjusted so as to maintain a forever balanced budget? No! As the above example as well as business cycle theory suggest, maintaining tax rates constant ("tax smoothing"), and thereby allowing government deficits and surpluses to arise, will generally make more sense. In itself, a budget deficit is not worrisome. It only becomes worrisome if it is not accompanied later by sufficient budget surpluses to avoid an exploding government debt/GDP ratio to arise. This requires that the tax rates taken together have a *level* which in the long run matches the level of government expenses.

11.6 Exercises

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