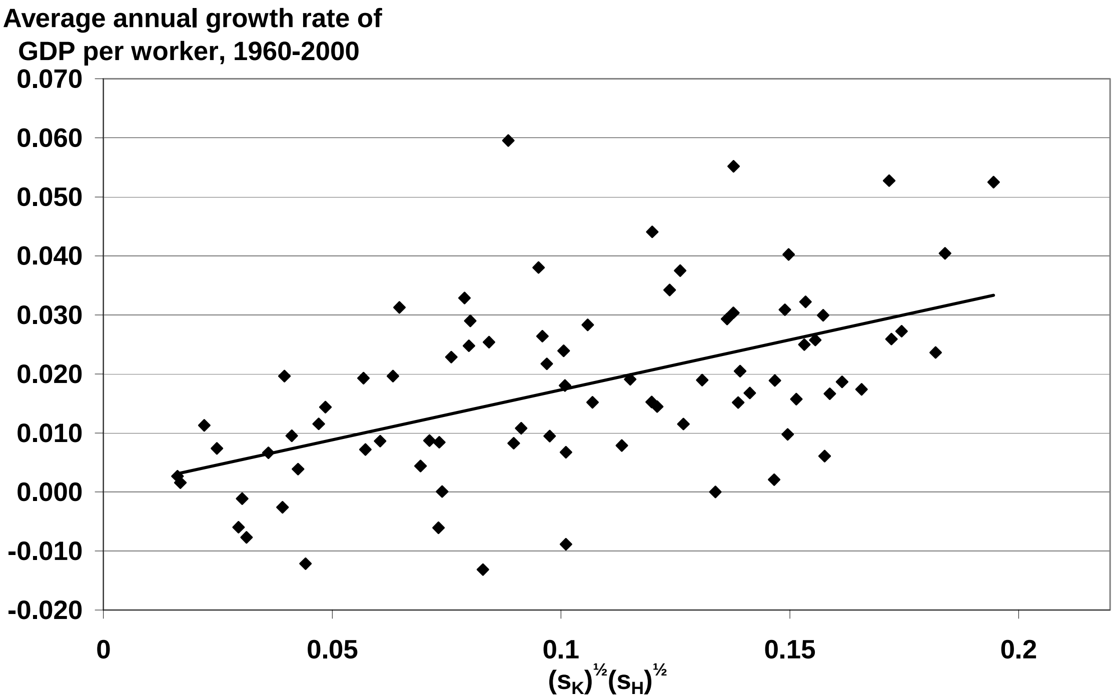

Exercise 8.7. Semi-endogenous growth and endogenous growth if the productive externality arises from $\mathbf{Y}_t$ rather than from $\mathbf{K}_t$

Note the typo in this exercise *in the first print of the book*. The transition equation in Question 3 should read:

$$\begin{aligned}\tilde{k}_{t+1} &= \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \tilde{k}_t \left(s\tilde{k}_t^{\alpha-1} + (1-\delta)\right)^{\frac{1-\phi}{1-\phi(1-\alpha)}} \\ &= \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \left(s\tilde{k}_t^{\frac{\alpha}{1-\phi}} + (1-\delta)\,\tilde{k}_t^{\frac{1-\phi(1-\alpha)}{1-\phi}}\right)^{\frac{1-\phi}{1-\phi(1-\alpha)}}.\end{aligned}$$

That is, the $\tilde{k}_t$ on the left hand side should be replaced by $\tilde{k}_{t+1}$, and the $s\tilde{k}_t^{[\alpha-\phi(1-\alpha)]/(1-\phi)}$ inside the parenthesis in the second line should be replaced by $s\tilde{k}_t^{\alpha/(1-\phi)}$.

1. The aggregate production function is found by inserting $A_t = Y_t^{\phi}$ into (6):

$$Y_t = (K_t)^{\alpha} \left(Y_t^{\phi} L_t\right)^{1-\alpha} \Leftrightarrow$$

$$Y_t^{1-\phi(1-\alpha)} = K_t^{\alpha} L_t^{1-\alpha} \Leftrightarrow$$

$$Y_t = K_t^{\frac{\alpha}{1-\phi(1-\alpha)}} L_t^{\frac{1-\alpha}{1-\phi(1-\alpha)}}.$$

Assuming that $\phi < 1/(1-\alpha)$ ensures that the exponents on K_t and L_t are positive. The sum of the exponents in the aggregate production function is: $\frac{\alpha}{1-\phi(1-\alpha)} + \frac{1-\alpha}{1-\phi(1-\alpha)} = \frac{1}{1-\phi(1-\alpha)} > 1$, so this function exhibits increasing returns to K_t and L_t. When $\phi = 1$ the production function reduces to

$$Y_t = K_t L_t^{\frac{1-\alpha}{\alpha}},$$

which has constant returns to K_t alone.

2. From $A_t = Y_t^{\phi}$ and the aggregate production function one gets:

$$\frac{A_{t+1}}{A_t} = \frac{Y_{t+1}^{\phi}}{Y_t^{\phi}} = \left(\frac{K_{t+1}}{K_t}\right)^{\frac{\alpha\phi}{1-\phi(1-\alpha)}} \left(\frac{L_{t+1}}{L_t}\right)^{\frac{\phi(1-\alpha)}{1-\phi(1-\alpha)}}.$$

3. Parallel to the chapter's analysis of the model of semi-endogenous growth (Section 2) one derives:

$$\begin{aligned}
\frac{\tilde{k}_{t+1}}{\tilde{k}_t} &= \frac{\frac{K_{t+1}}{K_t}}{\frac{A_{t+1}}{A_t}\frac{L_{t+1}}{L_t}} = \frac{\frac{K_{t+1}}{K_t}}{\left(\frac{K_{t+1}}{K_t}\right)^{\frac{\alpha\phi}{1-\phi(1-\alpha)}} \left(\frac{L_{t+1}}{L_t}\right)^{\frac{\phi(1-\alpha)}{1-\phi(1-\alpha)}} \frac{L_{t+1}}{L_t}} \\
&= \frac{\left(\frac{K_{t+1}}{K_t}\right)^{1-\frac{\alpha\phi}{1-\phi(1-\alpha)}}}{\left(\frac{L_{t+1}}{L_t}\right)^{\frac{\phi(1-\alpha)}{1-\phi(1-\alpha)}+1}} = \frac{\left(\frac{K_{t+1}}{K_t}\right)^{\frac{1-\phi}{1-\phi(1-\alpha)}}}{\left(\frac{L_{t+1}}{L_t}\right)^{\frac{1}{1-\phi(1-\alpha)}}} \\
&= \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \left(\frac{K_{t+1}}{K_t}\right)^{\frac{1-\phi}{1-\phi(1-\alpha)}}.
\end{aligned}$$

Now using the capital accumulation equation gives:

$$\begin{aligned}
\frac{\tilde{k}_{t+1}}{\tilde{k}_t} &= \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \left(\frac{sY_t + (1-\delta)K_t}{K_t}\right)^{\frac{1-\phi}{1-\phi(1-\alpha)}} \\
&= \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \left[\frac{s\tilde{y}_t}{\tilde{k}_t} + (1-\delta)\right]^{\frac{1-\phi}{1-\phi(1-\alpha)}}.
\end{aligned}$$

Inserting $\tilde{y}_t = \tilde{k}_t^{\alpha}$, which follows from the production function, gives:

$$\frac{\tilde{k}_{t+1}}{\tilde{k}_t} = \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \left[s\tilde{k}_t^{\alpha-1} + (1-\delta)\right]^{\frac{1-\phi}{1-\phi(1-\alpha)}} \Leftrightarrow$$

$$\begin{aligned}
\tilde{k}_{t+1} &= \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \tilde{k}_t \left[s\tilde{k}_t^{\alpha-1} + (1-\delta)\right]^{\frac{1-\phi}{1-\phi(1-\alpha)}} \\
&= \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \left(s\tilde{k}_t^{\frac{\alpha}{1-\phi}} + (1-\delta)\,\tilde{k}_t^{\frac{1-\phi(1-\alpha)}{1-\phi}}\right)^{\frac{1-\phi}{1-\phi(1-\alpha)}}.
\end{aligned}$$

The last formula gives two alternative expressions for the transition curve.

By definition, in steady state $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}^*$. Insert this in the transition equation (first expression) to find the steady state value of $\tilde{k}_t$:

$$1 = \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \left(s\left(\tilde{k}^*\right)^{\alpha-1} + (1-\delta)\right)^{\frac{1-\phi}{1-\phi(1-\alpha)}} \Leftrightarrow$$

$$(1+n)^{\frac{1}{1-\phi(1-\alpha)}} = \left(s\left(\tilde{k}^*\right)^{\alpha-1} + (1-\delta)\right)^{\frac{1-\phi}{1-\phi(1-\alpha)}} \Leftrightarrow$$

$$(1+n)^{\frac{1}{1-\phi}} = s\left(\tilde{k}^*\right)^{\alpha-1} + (1-\delta) \Leftrightarrow$$

$$\tilde{k}^* = \left(\frac{s}{(1+n)^{\frac{1}{1-\phi}} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}.$$

Using $\tilde{y}_t = \tilde{k}_t^{\alpha}$ gives the steady state value of $\tilde{y}_t$:

$$\tilde{y}^* = \left(\tilde{k}^*\right)^{\alpha} = \left(\frac{s}{(1+n)^{\frac{1}{1-\phi}} - (1-\delta)}\right)^{\frac{\alpha}{1-\alpha}}.$$

Under the stated condition, $(1+n)^{\frac{1}{1-\phi}} > (1-\delta)$, the denominators above are positive and the expressions for $\tilde{k}^*$ and $\tilde{y}^*$ are meaningful.

It has thus been established that the transition equation has a unique strictly positive intersection with the 45°-line. Furthermore, the transition curve passes through $(0,0)$ and is everywhere strictly increasing, as can be verified directly from inspection of the second formula for the transition curve (note that $\phi < 1$ implies that all the exponents are positive). If the slope of the transition curve at the intersection with the 45°-line is smaller than one, convergence to steady state follows from 'stair case iteration' in the usual transition diagram. We compute the derivative (using the first of the expressions for the transition curve:

$$\frac{d\tilde{k}_{t+1}}{d\tilde{k}_t} = \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \cdot$$

$$\left[\left[s\tilde{k}_t^{\alpha-1} + (1-\delta)\right]^{\frac{1-\phi}{1-\phi(1-\alpha)}} + \frac{(1-\phi)(\alpha-1)}{1-\phi(1-\alpha)}\left[s\tilde{k}_t^{\alpha-1} + (1-\delta)\right]^{\frac{1-\phi}{1-\phi(1-\alpha)}-1} s\tilde{k}_t^{\alpha-1}\right]$$

$$= \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \cdot \left[s\tilde{k}_t^{\alpha-1} + (1-\delta)\right]^{\frac{1-\phi}{1-\phi(1-\alpha)}-1}\left[s\tilde{k}_t^{\alpha-1} + (1-\delta) - \frac{(1-\phi)(1-\alpha)}{1-\phi(1-\alpha)} s\tilde{k}_t^{\alpha-1}\right].$$

Inserting here our expression for $\tilde{k}^*$ in place of $\tilde{k}_t$ gives the slope at $\tilde{k}^*$:

$$\left.\frac{d\tilde{k}_{t+1}}{d\tilde{k}_t}\right|_{\tilde{k}_t=\tilde{k}^*} =$$

$$\left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \cdot \left[(1+n)^{\frac{1}{1-\phi}}\right]^{\frac{1-\phi}{1-\phi(1-\alpha)}-1}\left[(1+n)^{\frac{1}{1-\phi}} - \frac{(1-\phi)(1-\alpha)}{1-\phi(1-\alpha)}\left((1+n)^{\frac{1}{1-\phi}} - (1-\delta)\right)\right]$$

$$= \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi(1-\alpha)}} \cdot \left(\frac{1}{1+n}\right)^{\frac{1}{1-\phi}\cdot\frac{\alpha\phi}{1-\phi(1-\alpha)}} \cdot \left(\frac{1}{1+n}\right)^{-\frac{1}{1-\phi}} \cdot$$

$$\left[1 - \frac{(1-\phi)(1-\alpha)}{1-\phi(1-\alpha)}\left(1 - \frac{1-\delta}{(1+n)^{\frac{1}{1-\phi}}}\right)\right]$$

$$= 1 - \frac{(1-\phi)(1-\alpha)}{1-\phi(1-\alpha)}\left(1 - \frac{1-\delta}{(1+n)^{\frac{1}{1-\phi}}}\right).$$

This is positive and smaller than one since: $(1+n)^{\frac{1}{1-\phi}} > (1-\delta)$ implies that the parenthesis is positive and smaller than one. The factor in front of the parenthesis is itself positive and smaller than one, so the product is positive and smaller than one. One minus the product must then also be positive and smaller than one.

4. Since $\tilde{y}_t \equiv y_t/A_t$ is constant in steady state, y_t must grow at the same rate as A_t, that is, $y_{t+1}/y_t = A_{t+1}/A_t$. From $A_t = Y_t^{\phi}$:

$$\frac{A_{t+1}}{A_t} = \left(\frac{Y_{t+1}}{Y_t}\right)^{\phi} = \left(\frac{y_{t+1}}{y_t}\right)^{\phi}(1+n)^{\phi}.$$

Inserting $A_{t+1}/A_t = y_{t+1}/y_t$ (which holds *in steady state*) and rearranging gives:

$$\frac{y_{t+1}}{y_t} = \left(\frac{y_{t+1}}{y_t}\right)^{\phi}(1+n)^{\phi} \iff \frac{y_{t+1}}{y_t} = (1+n)^{\frac{\phi}{1-\phi}} \Leftrightarrow$$

$$\frac{y_{t+1} - y_t}{y_t} = (1+n)^{\frac{\phi}{1-\phi}} - 1.$$

When $\phi < 1$ the exponent on $1+n$ is positive. Hence a larger population growth rate gives a higher growth rate of output per worker in steady state.

5. At the end of Question 1 we found that:

$$Y_t = K_t L_t^{\frac{1-\alpha}{\alpha}},$$

in case of $\phi = 1$. Setting $n = 0$ removes the time subscript on L_t. Define $A \equiv (L)^{(1-\alpha)/\alpha}$. Then:

$$Y_t = AK_t.$$

Since there is no population growth and the productive externality is already embedded in the aggregate production function, the equation above and the capital accumulation equation (8),

$$K_{t+1} = sY_t + (1-\delta)K_t,$$

make up the entire model. Inserting $Y_t = AK_t$ into the capital accumulation equation and rearranging gives the growth rate of capital:

$$K_{t+1} = sAK_t + (1-\delta)\,K_t \Leftrightarrow \frac{K_{t+1} - K_t}{K_t} = sA - \delta,$$

which, since $Y_t = AK_t$ and there is a constant labour force, is also the growth rate of Y_t, k_t, and y_t. We have, of course, found the same growth rate as in the chapter's 'AK-model', only with a slight difference in the definition of A. All features with respect to policy implications, scale effects etc. therefore bear over from the chapter's model to the one considered here.

Exercise 8.8. Taxation and productive government spending: endogenous growth without productive externalities