Exercises in Growth Theory and Empirics*

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Exercise 1: Empirical implications of the Solow Model

Consider the following growth model for a closed economy:

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \tag{1}$$

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t) \tag{2}$$

$$C(t) = (1-s)Y(t) \tag{3}$$

$$\dot{A}(t)/A(t) = x \tag{4}$$

$$\dot{L}(t)/L(t) = n. \tag{5}$$

The notation is as follows: $Y(t)$ is output, $K(t)$ is the stock of physical capital, $A(t)$ represents "technology", $C(t)$ is consumption. Its assumed that $0 < \alpha < 1, \delta > 0, s > 0, x > 0$ and $n > 0$ are constant over time

Question 1.

(a) Show that the system (1) - (5) implies that income per efficiency units of labor ($y \equiv Y/AL$) evolves according to

$$\dot{y}(t)/y(t) = \alpha\left(sy(t)^{-\frac{1-\alpha}{\alpha}} - (n+\delta+x)\right). \tag{6}$$

(*Hint*: Derive the law of motion for capital per efficiency units of labor. Next use that income per efficiency units of labor is related to capital per efficiency units of labor in the following way $y(t) = k(t)^{\alpha}$).

(b) Provide an economic interpretation of this equation – in particular why $\dot{y}(t)/y(t)$ depend on $y(t)$ in the manner indicated.

(c) Solve for the steady state level of income per efficiency units of labor y^*.

*Developed for the course "economic growth", Spring 2003.

Question 2.

Show that a log-linearization of equation (6) around steady state ($y = y^*$) yields

$$\dot{y}(t)/y(t) \approx \lambda \cdot (\ln y^* - \ln y(t)), \tag{7}$$

where

$$\lambda \equiv (1-\alpha)(n+\delta+x).$$

Question 3

(a) What's the interpretation of λ? (b) Explain why the various parameters matter for λ in the manner indicated by the formula.

Question 4.

Assume s rises by one percentage point. What would be the *impact* effect on growth in income per capita ($= \dot{y}/y + x$), assuming we are in the vicinity of steady state and under plausible assumptions regarding various parameters?

Question 5.

Show that the differential equation (7) can be solved to yield

$$\ln y(t) = \ln y(0)\, e^{-\lambda t} + \ln y^* \left(1 - e^{-\lambda t}\right). \tag{8}$$

Question 6.

Show that equation (8) along with the expression for y^* and the added assumption that

$$\ln A(0) = \ln \bar{A}$$

implies:

$$\ln\left(\frac{Y(t)}{L(t)}\right) - \ln\left(\frac{Y(0)}{L(0)}\right) = \beta_0 + \beta_1 \ln \frac{Y(0)}{L(0)} + \beta_2 \ln\left(\frac{s}{n+\delta+x}\right).$$

where $\beta_0 = \left[xt + \ln \bar{A}\left(1 - e^{-\lambda t}\right)\right]$, $\beta_1 = -\left(1 - e^{-\lambda t}\right)$, $\beta_2 = \left(1 - e^{-\lambda t}\right)\frac{\alpha}{1-\alpha}$.

Question 7.

The table below shows the results one obtains if the above equation is estimated by OLS on a cross section of countries. (a) Comment on how these results differ from what you would expect (you may concentrate on the results from the "intermediate sample"). (b) Provide possible explanations for such differences.

TABLE IV

TESTS FOR CONDITIONAL CONVERGENCE

Dependent variable: log difference GDP per working-age person 1960–1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	1.93	2.23	2.19
	(0.83)	(0.86)	(1.17)
ln(Y60)	−0.141	−0.228	−0.351
	(0.052)	(0.057)	(0.066)
ln(I/GDP)	0.647	0.644	0.392
	(0.087)	(0.104)	(0.176)
$\ln(n + g + \delta)$	−0.299	−0.464	−0.753
	(0.304)	(0.307)	(0.341)
$\bar{R}^2$	0.38	0.35	0.62
s.e.e.	0.35	0.33	0.15
Implied λ	0.00606	0.0104	0.0173
	(0.00182)	(0.0019)	(0.0019)

Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05.

Source: Mankiw, Romer and Weil (1992).

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EXERCISE 2: CONVERGENCE

Consider the following statement:

> "In neoclassical growth models ... a country's growth rate tends to be inversely related to its starting level of income per capita...Thus there is a force that promotes convergence in per capita income levels across countries. The main element behind the convergence result in neoclassical growth models is diminishing returns to reproducible capital."

Source: R. Barro, 1991, Economic Growth in a Cross-Section of countries. *Quarterly Journal of Economics*, 106, p. 407.

Question:

Do you necessarily agree with this statement? Explain why or why not.

Exercise 3: Growth Accounting

The Table below shows the result from two growth accounting studies on various South-East Asian countries. As can be seen, there are major differences when comparing so-called "primal" results, to "dual" results.

Question:

Describe the two methodologies, and explain why the two approaches may yield different results.

		Annual growth rate of:			
Real interest rate	Labor share	Rental price of capital	Wages	Dual TFP	Primal TFP
Singapore:					
Return on equity (1971–1990)	0.51	0.09	3.13	1.64	−0.69
Average lending rate (1968–1990)	0.51	1.21	2.69	1.96	−0.30
E/P ratio (1972–1990)	0.51	1.27	3.46	2.39	−0.68
Taiwan:					
Informal loan rate (1966–1990)	0.74	−0.75	5.26	3.72	2.10
Deposit rate (1966–1990)	0.74	−0.77	5.26	3.71	2.10
Secured loan rate (1966–1990)	0.74	−1.73	5.26	3.46	2.10
Treasury-bill rate (1973–1990)	0.75	−1.52	5.24	3.52	2.06
Hong Kong:					
Best lending rate (1966–1991)	0.63	0.29	4.04	2.65	2.30
Call-money rate (1966–1991)	0.63	−0.65	4.04	2.30	2.30
E/P ratio (1973–1991)	0.62	−0.42	4.18	2.41	2.18
Korea:					
Curb-market loan rate (1966–1990)	0.70	−4.84	4.38	1.64	1.70
Deposit rate (1966–1990)	0.70	−3.88	4.38	1.93	1.70
Discount rate (1966–1990)	0.70	−3.33	4.38	2.09	1.70

Notes: Dual TFPG is the weighted growth rate of quality-adjusted real wages and rental price of capital, where the weights are the factor shares. Primal TFPG and aggregate factor shares are calculated from Young (1995). The return on equity and earnings–price ratio in Singapore and the earnings–price ratio in Hong Kong are used as *real* interest rates. All other measures of the return to capital are used as nominal interest rates from which the *ex post* inflation rate is subtracted to obtain the real interest rate. See Hsieh (1998) for further details.

Exercise 4: Growth accounting

Assume that the Danish economy can be described by the following aggregate production function

$$Y = K^{\alpha} (AL)^{1-\alpha},$$

and let $\dot{A}/A = x$, while $\dot{L}/L = n$. While n is observable, x can only be inferred from the data using the following equation:

$$\frac{\dot{Y}}{Y} - \alpha \frac{\dot{K}}{K} - (1-\alpha) \frac{\dot{L}}{L} = (1-\alpha) \frac{\dot{A}}{A}$$

Assume that $x = 0.01$, $n = 0.01$, $\alpha = 1/3$ and that Denmark, over the period under consideration, is in a steady state.

Question 1

Denote the growth contribution of factor z by:

$$\psi_z \equiv \frac{\text{growth accounting contribution of } z}{\dot{Y}/Y}$$

What fraction of $\dot{Y}/Y$ can be accounted for by capital, human input and TFP, respectively, under the growth accounting framework?

In the coming years the growth rate of the labor force will be declining, as the population of the Danish economy ages. Specifically, its known that n will decrease to zero.

Question 2

A politician has read you analysis above, and makes the following inference: "*Due to declining growth of the labor force, one can expect that the GDP of Denmark only will grow by* $(1-\psi_L)\cdot \dot{Y}/Y$ *percent per year, in the years to come*". Do you agree with this inference?

Next consider Sweden. As it turns out, Sweden and Denmark has the same α, based on national accounts data, and both economies are well described by the Cobb-Douglas production function above. The difference is, however, that Sweden is not in steady state. In fact, Sweden is converging towards steady state from below.

Question 3

If you were to perform a growth accounting study on the Swedish economy, would you expect ψ_K to be higher or lower than in Denmark? (*Hint:* Think about what is happening to the capital-output ratio as the economy converges to steady state from below. A geometric argument is sufficient)

Exercise 5: Convergence

Consider the following production function

$$Y = B\left(\tilde{k}\right) K^{\alpha} (AL)^{1-\alpha},$$

where

$$B\left(\tilde{k}\right) = \begin{cases} \bar{B} \text{ if } \tilde{k} \geq \phi \\ \underline{B} \text{ if } \tilde{k} < \phi \end{cases}.$$

Otherwise the model follows the assumptions of a standard Solow model (cf. exercise 1).

Question 1

Show that the dynamic system, expressed in capital in efficiency units of labor, is given by

$$\dot{\tilde{k}} = \begin{cases} s\bar{B}\tilde{k}^{\alpha} - (n+\delta+x) & \text{if } \tilde{k} \geq \phi \\ s\underline{B}\tilde{k}^{\alpha} - (n+\delta+x) & \text{if } \tilde{k} < \phi \end{cases}.$$

Question 2

Does this model necessarily imply multiple steady states in $\tilde{k}$? A geometrical argument is sufficient.

Question 3

Assume multiple steady states *do* arise in the model above. Show that the model, if log-linearized in the vicinity of the two steady states we get the following law of motion for $Y/L = y(t)$

$$\ln y(t) - \ln y(0) = \begin{cases} \beta_0^h + \beta_1 \ln y(0) + \beta_2 \ln\left(\frac{s}{n+\delta+x}\right) & \text{for } \tilde{k}(0) \geq \phi \\ \beta_0^l + \beta_1 \ln y(0) + \beta_2 \ln\left(\frac{s}{n+\delta+x}\right) & \text{for } \tilde{k}(0) < \phi \end{cases}. \tag{9}$$

where $\beta_0^h \equiv xt + \left(1-e^{-\lambda t}\right)\ln A(0) + \frac{1-e^{-\lambda t}}{1-\alpha}\ln\bar{B}$, $\beta_0^l \equiv xt + \left(1-e^{-\lambda t}\right)\ln A(0) + \frac{1-e^{-\lambda t}}{1-\alpha}\ln\underline{B}$, $\beta_1 = -\left(1-e^{-\lambda t}\right)$ and $\beta_2 = -\beta_1\frac{\alpha}{1-\alpha}$.

Assume that the "world" is characterized by the above dynamics. Nevertheless, an uninformed researcher estimates the following equation by OLS:

$$\ln y(t) - \ln y(0) = \beta_0 + \beta_1 \ln y(0) + \beta_2 \frac{\alpha}{1-\alpha}\ln\left(\frac{s}{n+\delta+x}\right). \tag{10}$$

Accordingly he believes β_0 is the same across countries, and that β_1 contains information on the rate of convergence λ. Another researcher has in fact figured out that "the world is described" by equation (9), and therefor splits his data into two groups separated by ϕ (which she claims to have identified).

Let λ^a be the OLS estimate for the rate of convergence obtained by the first researcher, while λ^b is the estimate the second researcher obtains.

Question 4

Would you expect $\lambda^a \gtreqless \lambda^b$? Explain.

Now we lower a wail of ignorance. We no longer know whether equation (9) or (10) represents "the truth". But it *is* the case that splitting our large sample of country observations into sub-groups, where upon we estimate equation (10) on each of these, then our estimate for λ changes in the manner deduced above.

Question 5

Is this conclusive evidence in favor of club convergence? *Explain*: Why is it or why isn't it?