Exercises in Growth Theory and Empirics*

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Exercise 10: endogenous policy and endogenous growth

Consider an economy inhabited by L *infinitely lived agents indexed by* i*. The size of population is constant through time. It is assumed that agents differ with respect to their initial endowment of capital,* $K^i(0)$*. In all other respects, agents are identical. The government obtains its revenue from taxing wealth, at the rate* τ*. Ultimately,* τ*, is determined by majority voting. The proceeds from taxing wealth are used to finance lump sum transfers to all* L *agents at the rate* θ*. The government balances its budget. Hence*

$$\theta(t)L = \tau(t)\sum_{i}^{L} k^i(t) = \tau(t)k(t)L,$$

where $k(t) \equiv K(t)/L$ *is income per capita. The tax rate is assumed to be time invariant. Hence*

$$\tau(t) = \tau \ \forall t.$$

The average capital stock in the economy is defined as $k(t) \equiv K(t)/L$*.*

The problem facing agent i *is to*

$$\max_{\{c^i(t)\}_{t=0}^{\infty}} \int_0^{\infty} \ln c^i(t)\, e^{-\rho t} dt, \ \ \rho > 0$$

$$c^i(t) \geq 0,$$

$$\dot{k}^i(t) = (r-\tau)k^i + \theta(t) - c^i(t), \ \ k^i(0) \ \text{given},$$

$$\lim_{t\to\infty} k^i(t)e^{-rt} \geq 0.$$

The aggregate production function is of the AK-variety. Hence, at all points in time $r = A$*.*

Question 1.

Solve the above maximization problem, and show that

$$\frac{\dot{c}^i(t)}{c^i(t)} = A - \tau - \rho.$$

Comment on this first order condition.

Given the AK-structure of the model it holds that

$$\gamma \equiv \frac{\dot{c}^i(t)}{c^i(t)} = A - \tau - \rho = \frac{\dot{k}^i(t)}{k^i(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{\dot{\theta}(t)}{\theta(t)} \ \forall i.$$

Question 2.

Use this result to establish that

$$c^i(t) = \left[\tau\sigma^i + \rho\right] k^i(0)\, e^{\gamma t},$$

where $\sigma^i \equiv k(t)/k^i(t)$ and constant through time.

Question 3.

The problem of choosing the preferred tax rate for individual i is

$$\max_{\tau} \int_0^{\infty} \ln c^i(t)\, e^{-\rho t} dt, \ \ \rho > 0$$

s.t.

$$c^i(t) = \left[\tau\sigma^i + \rho\right] k^i(0)\, e^{\gamma t}$$
$$\gamma = A - \tau - \rho.$$

Show that this problem is equivalent to the static maximization problem

$$\max_{\tau} \frac{1}{\rho}\left(\ln c^i(0) + \frac{\gamma}{\rho}\right),$$

s.t.

$$c^i(0) = \left[\tau\sigma^i + \rho\right] k^i(0).$$

(*Hint:* Show that $\int_0^\infty \ln\left(c^i(0)\, e^{\gamma t}\right) e^{-\rho t} dt = \frac{1}{\rho}\left(\ln c^i(0) + \frac{\gamma}{\rho}\right)$. To do so, you'll find it nessesary to invoke the following formula for integration by parts: $\int_a^b f(x)\, g'(x)\, dx = |_a^b\, (f(x)\, g(x)) - \int_a^b f'(x)\, g(x)$.)

Question 4.

(i) Solve the problem for individual i of choosing the preferred tax rate τ^i. (ii) Explain why τ^i depends on σ^i, A and ρ in the manner indicated by the formula. (iii) Assuming majority voting, and full participation at elections, what will be the implemented tax rate?

Question 5.

Derive the long-run growth rate of the economy. What is the implied relationsship between γ and skewness of the income distribution of capital?

Exercise 11: human capital and real rate of interest differences[1]

Consider a closed economy, where output, $Y(t)$, is produced using the following constant returns technology $Y(t) = K(t)^\alpha L(t)^{1-\alpha}$ where $K(t)$ is capital input and $L(t)$ is the labor force. Firm's operate in competitive markets and maximize profits. Specifically, at all points in time it hold that

$$r = \alpha \frac{Y}{K},$$

where r is the real rate of return.

In 1990 income per worker, $Y(t)/L(t)$, of the United States was 15 times the comparable number for India.

[1]This exercise draws on R. Lucas Jr., 1990. "Why Doesn't Capital Flow to Poor Countries?" *American Economic Review Papers and Proceeding*, May, pp. 92-96.

1. Assuming that $\alpha = 0.4$, what is the implied difference in real rate of return?

In an influencial article, Robert Lucas argued that the puzzle could be resolved if one recognized that there are systematic differences in labor productivity attributable to human capital. Specifically, suppose we were to consider the following formulation for the production function:

$$Y(t) = K(t)^{\alpha} (h(t) L(t))^{1-\alpha} \bar{h}(t)^{\gamma},$$

where h is an index representing human capital of the representative worker, while the term $\bar{h}(t)^{\gamma}$ represents an externality from the average level of human capital accumulation. It follows that in equilibrium $h = \bar{h}$. Lucas argues that it is reasonable to assume that $\gamma \approx 0.4$. Moreover, based on available evidence, Lucas' assess that the level of human capital in the US, in 1990, was about 5 times the level in India, i.e. $h^{US}/h^{India} = 5$.

3. Given these assumptions. What is the implied difference in r? Provide the intuition behind the difference in results compared with the scenario without human capital.

4 Is $\gamma \approx 0.4$ empirical plausible in light of recent empirical evidence on the productive effects of human capital?