Endogenous growth through R&D

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Lecture notes

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Final goods sector

Firm is technology

$$Y_i = AL_i^{1-\alpha} \sum_j^N x_{ij}^\alpha$$

assume for a moment $x_{ij} = x_i \Rightarrow Y_i = AL_i^{1-\alpha} x_i^\alpha N$. Observe

1. CRTS in N = # int. goods = # "ideas". N$\uparrow$ could be thought to capture specialization.

2. Increasing x would not allow for perpetual growth. Diminishing returns

3. CRTS in x and L (rival inputs)

Final goods sector (Cont'ned)

The max problem is to choose

$$\left\{\{x_{ji}\}_{j=1}^{N}, L_i\right\} = \arg\max AL_i^{1-\alpha}\sum_{j}^{N} x_{ij}^{\alpha} - wL_i - \sum^{N} p_j x_{ij}$$

FOC

$$(1-\alpha)\frac{Y_i}{L_i} = w$$

$$\alpha A L_i^{1-\alpha} x_{ij}^{\alpha-1} = p_j \text{ for all } j$$

Notice the demand for the jth intermediate good is the same across all i. So aggregate demand for $j$the intermediate good

$$x_j^d = \sum_i x_{ij} = \sum \left(\frac{\alpha A}{p_j}\right)^{\frac{1}{1-\alpha}} L_i = \left(\frac{\alpha A}{p_j}\right)^{\frac{1}{1-\alpha}} \underbrace{\sum_i L_i}_{=L}$$

Intermediate good sector

Use 1 unit of output to produce 1 unit of the intermediate good

$$\max_{p_j, x_j} \left(p_j - 1\right) x_j$$

s.t $\alpha A L^{1-\alpha} x_j^{\alpha-1} = p_j$.

Total revenue:

$$p_j x_j = \alpha A L^{1-\alpha} x_j^{\alpha}$$

So

$$MR = \alpha^2 A L^{1-\alpha} x_j^{\alpha-1} = MC = 1$$

Hence, optimal quantity

$$x_j = \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L = \bar{x} \text{ for all } j$$

Intermediate good sector (Cont'ned)

Monopoly price (substitute into demand curve)

$$p_j = \alpha A L^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L \right)^{\alpha - 1} = \alpha^{-1} = \bar{p} \text{ for all } j.$$

Observe $p > MC$, so α basically paramterizes the mark-up.

Profits:

$$\pi_j = (p_j - 1)\, x_j = \left(\frac{1}{\alpha} - 1 \right) \bar{x} = \bar{\pi}$$

Finally, total value of a patent

$$V(t) = \int_t^\infty \bar{\pi} e^{-\int_t^v r_\tau d\tau} dv$$

R&D sector

Aggregate production technology

$$\dot{N} = Y^R/\eta$$

or: using 1 unit of output creates $1/\eta$ ideas.

Now, we could define $\sigma_R Y = Y^R$. Then we have

$$\dot{N} = [\sigma_R/\eta]\, Y$$

which is basically a "MRW" assumption; now for R&D. "Lab equipment" approach. Also, note that given symmetrical equilibrium:

$$\dot{N} = [\sigma_R/\eta]\, Y = [\sigma_R/\eta]\, AL^{1-\alpha} N\bar{x}^{\alpha}.$$

Another way to think about the assumption: $R\&D$ uses σ^R fraction of economy's resources (L,x)

R&D sector (cont'ned)

To produce 1 idea, use η units of output.

Value of 1 idea: $V(t)$.

For equilibrium $V(t) = \eta$. (suppose otherwise)

Given this

$$\eta = V(t) = \int_t^\infty \bar{\pi} e^{-\int_t^v r_\tau d\tau} dv$$

which implies $r_\tau = r$. Thus (solving the integral)

$$\frac{\bar{\pi}}{\eta} = r,$$

the equilibrium return. Complete solution for the real rate of interest

$$r = \frac{\bar{\pi}}{\eta} = \frac{1-\alpha}{\alpha\eta} \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L.$$

Households

$$\max \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} L e^{-\rho t}$$

s.t.

$$c(t) \geq 0,$$

$$\dot{b} = rb + w - c, b_0 \text{ given},$$

$$\lim_{t\to\infty} b_t e^{-rt} \geq 0.$$

Standard problem, leads to the FOC

$$\dot{c}/c = \frac{1}{\theta}(r-\rho), \quad \lim_{t\to\infty} b_t \lambda_t = 0$$

Equilibrium

First, note that we can aggregate so as to get GDP

$$Y_i = AL_i^{1-\alpha}\sum_j^N x_{ij}^\alpha \underset{\text{sym. eq}}{=} AL_i^{1-\alpha}N\bar{x}_i^\alpha$$
$$= A\left(\frac{L_i}{\bar{x}_i}\right)^{-\alpha} NL_i$$

since all firms face same factor prices w, p_j and use the same technology $\left(\frac{L_i}{\bar{x}_i}\right)^{-\alpha}$ is the same

$$Y = \sum_i Y_i = A\left(\frac{L_i}{\bar{x}_i}\right)^{-\alpha} N\sum L_i = A\bar{x}^\alpha NL^{1-\alpha}$$

Note: "AK model". No transitional dynamics + balanced growth.

Equilibrium

The Keynes-Ramsey rule pins down the growth rate then

$$\gamma = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}\left(\frac{\bar{\pi}}{\eta}-\rho\right) = \frac{1}{\theta}\left(\frac{1}{\eta}\frac{1-\alpha}{\alpha}\bar{x}-\rho\right)$$

where $\bar{x} = \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L$.

Hence, at any given point in time, income per capita

$$\begin{aligned} Y/L &= A\bar{x}^{\alpha} L^{-\alpha} N \\ &= A\left(\alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L\right)^{\alpha} L^{-\alpha} N \\ Y/L &= A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} N(0) e^{\gamma t} \end{aligned}$$

Observations: (1) $\eta \uparrow \Rightarrow \gamma \downarrow$. (2) $L \uparrow \Rightarrow \gamma \uparrow$.

Planner problem

$$\max \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} L e^{-\rho t}$$

s.t.

$$c(t) \geq 0, x \geq 0$$

the resource constraint

$$Y = cL + Nx + \eta \dot{N} \Leftrightarrow \dot{N} = \frac{1}{\eta} [Y - cL - Nx]$$

aggregate production function

$$Y = AL^{1-\alpha} x^\alpha N$$

$$N \geq 0 \text{ for all } t.$$

Planner problem (cont'ned)

Hamiltonian:

$$H(c,x,N,\lambda,t)=\frac{c_t^{1-\theta}}{1-\theta}Le^{-\rho t}+\lambda\frac{1}{\eta}[Y-cL-Nx]$$

$$c: c^{-\theta}Le^{-\rho t}=\frac{\lambda}{\eta}L$$

$$x: \frac{\lambda}{\eta}\left[\frac{\partial Y}{\partial x}-N\right]=0$$

$$N: \lambda\frac{1}{\eta}\left[\frac{\partial Y}{\partial N}-x\right]=-\dot{\lambda}$$

+TVC. As usual, we can use FOC wrt c,N to derive the K-R rule

$$\frac{\dot{c}}{c}=\frac{1}{\theta}\left[\frac{1}{\eta}\left(\frac{Y}{N}-x\right)-\rho\right]$$

Planner problem (cont'ned)

From the FOC wrt x : $\frac{\partial Y}{\partial x} - N = 0$, or

$$\frac{\partial Y}{\partial x} = \alpha \frac{Y}{x} = N \Leftrightarrow \frac{Y}{N} = \frac{x}{\alpha}$$

inserted into the K-R rule

$$\left(\frac{\dot{c}}{c}\right)^{sp} = \frac{1}{\theta}\left[\frac{1}{\eta}\left(\frac{1-\alpha}{\alpha}\right) x^{sp} - \rho\right]$$

structurally identical to the decentralized solution; x's level will differ however.

Use the production function

$$\frac{Y}{N} = AL^{1-\alpha} x^{\alpha} = AL^{1-\alpha}\left(\alpha \frac{Y}{N}\right)^{\alpha} \Leftrightarrow \frac{Y}{N} = A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} L$$

Finally, by $\frac{Y}{N} = \frac{x}{\alpha}$

$$x^{sp} = \alpha A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} L = A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L$$

Planner problem (cont'ned)

The full solution

$$\gamma^{sp} = \frac{1}{\theta}\left[\frac{1}{\eta}\left(\frac{1-\alpha}{\alpha}\right)x^{sp} - \rho\right]$$

where $x^{sp} = A^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}L$. Output

$$\begin{aligned}(Y/L)^{sp} &= A\left(x^{sp}\right)^{\alpha}L^{-\alpha}N = A\left(A^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}L\right)^{\alpha}L^{-\alpha}N \\ &= A^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}N\end{aligned}$$

the market solution

$$\gamma^{m} = \frac{1}{\theta}\left(\frac{1}{\eta}\frac{1-\alpha}{\alpha}\bar{x} - \rho\right)$$

where $\bar{x} = A^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}L$,

$$(Y/L)^{m} = A^{\frac{1}{1-\alpha}}\alpha^{2\cdot\frac{\alpha}{1-\alpha}}N$$

Planner problem (cont'ned): Comparison

Key thing to notice: $x^{sp} \cdot \alpha^{1/(1-\alpha)} = \bar{x}$.

So, since

$$r^m = \frac{1-\alpha}{\alpha}\frac{\bar{x}}{\eta}$$

it follows that $r^m < r^{sp}$, and therefore $\gamma^m < \gamma^{sp}$. In addition: the *level* is off as well.

1. Static monopoly distortion: To few intermediate goods are produced, $\bar{x} < x^{sp}$. Level of output too low.
2. Also implies too low a real rate of return; dynamic distortion.

Policy

Key thing to fix: $\bar{x} = x^{sp}$. This will ensure both static and dynamic efficiency since $\bar{x} = x^{sp} \Rightarrow r^m = r^{sp}, \gamma^m = \gamma^{sp}$ and in additon $(Y/L)^{sp} = (Y/L)^m$.

EX: subsidies final goods production.

Modified profit maximization problem

$$\left\{\{x_{ji}\}_{j=1}^{N}, L_i\right\} = \arg\max\,(1+\tau)\, AL_i^{1-\alpha} \sum_j^N x_{ij}^{\alpha} - wL_i - \sum^N p_j x_{ij}$$

where τ is the subsidy. Demand for good j

$$(1+\tau)\,\alpha x_{ij}^{\alpha-1} AL_i^{1-\alpha} = p_j.$$

As before, all firms face same demand, so aggregate demand:

$$(1+\tau)\,\alpha x_j^{\alpha-1} AL^{1-\alpha} = p_j.$$

Policy

Consider intermediate goods. Total revenue

$$(1+\tau)\,\alpha x_j^{\alpha} A L^{1-\alpha} = p_j x_j$$

Marginal revenue = marginal cost (optimal quantity)

$$MR = (1+\tau)\,\alpha^2 x_j^{\alpha-1} A L^{1-\alpha} = MC = 1$$

thus

$$x_j = \bar{x} = \left(\alpha^2 (1+\tau)\right)^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} L.$$

So $\bar{x} = x^{sp} = A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L$ obviously require

$$\alpha^2 (1+\tau) = \alpha \Rightarrow \tau = \frac{1-\alpha}{\alpha},$$

finance by lump sum taxes.

Alternative: Subsidize purchases of x_j : $(1+\tau)\, A L_i^{1-\alpha} \sum_j^N x_{ij}^{\alpha} - w L_i - (1-\tau) \sum^N p_j x_{ij}$. Here you'll find $\tau = (1-\alpha)$.

A Policy which doesn't work

R&D subsidy. Imagine you subsidize R&D outlays. So, in order to produce 1 idea, now requires $(1-\tau)\,\eta$ units of output. This means

$$r^m = \frac{\bar{\pi}}{\eta\,(1-\tau)} = \left(\frac{1-\alpha}{\alpha}\right)\frac{\bar{x}}{\eta\,(1-\tau)},$$

where $\bar{x} = \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L$. Clearly, we can choose τ such that $r^m = r^{sp} = \frac{1}{\eta}\left(\frac{1-\alpha}{\alpha}\right) A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L$.

But, the monopoly distortion is still there $\bar{x} < x^{sp}$.

As a result: $(Y/L)^m < (Y/L)^{sp}$. R&D subsidies fix the dynamic inefficiency, but not the static monopoly distortion.