The Rate of Convergence in the Solow model ... and then some*

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Abstract

This note does three things. First, we derive an estimation equation for the Solow model assuming we are in the vicinity of steady state and show briefly how important parameters can be obtained from the MRW study. As we go along we derive the rate of convergence to the steady state. Second, we provide details on Mankiw, Romer and Weil's back-of-the-envelope calculation to derive a prior for human capital's share in total income, i.e. "β". Third, we derive the rate of convergence in the augmented Solow model.

1 Estimating the Solow Model

We begin by noting that production in efficiency units, when the production function is Cobb-Douglas, can be written:

$$y = k^{\alpha}.$$

where $y \equiv Y/AL, k \equiv K/AL$. For brievity, we'll use the notation

$$\hat{x} \equiv \frac{\dot{x}}{x}$$

whenever we wish to denote a relative growth rate. Accordingly, the growth rate of income per efficiency unit of labor is:

$$\hat{y} = \alpha \hat{k}. \tag{1}$$

Now, the dynamical system governing capital in efficiency units is given by:

$$\dot{k} = sy - (n + \delta + x) k.$$

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Substituting this expression back into equation (1) yields

$$\hat{y} = \alpha \left(s \frac{y}{k} - (n + \delta + x) \right).$$

Next, we use that $y = k^{\alpha} \Leftrightarrow y^{1/\alpha} = k$. Inserting this fact into the equation above leads to

$$\begin{aligned} \hat{y} &= \alpha \left(s \frac{y}{y^{\frac{1}{\alpha}}} - (n + \delta + x) \right) \\ \hat{y} &= \alpha \left(s y^{-\frac{1-\alpha}{\alpha}} - (n + \delta + x) \right). \end{aligned} \tag{2}$$

This equation holds at all points in time, and indicate that, given structural charactaristics, richer countries will grow slower than initially poorer economies. While this is fine in general, we are preoccupied with deriving an equation for estimation purposes. And insofar as we would like to press the OLS bottom, then the equation better be linear.[1] This is the reason why we'll perform a *log*-linearization of the model around the steady state.

So here is a small trick. We note that

$$y = e^{\ln y}.$$

Using this fact in equation (2):

$$\hat{y} = \alpha \left(s e^{-\frac{1-\alpha}{\alpha} \ln y} - (n + \delta + x) \right) \equiv \phi (\ln y).$$

Now we perform the Taylor approximation:

$$\hat{y}_{\ln y = \ln y^*} \approx \phi (\ln y^*) + \phi\prime_{\ln y} (\ln y^*) (\ln y - \ln y^*) \tag{3}$$

First, $\phi (\ln y^*) = s (y^*)^{-\frac{1-\alpha}{\alpha}} - (n + \delta + x)$. In the steady state, this thing is zero. So

$$\phi (\ln y^*) = 0$$

Second term:

$$\begin{aligned} \phi\prime_{\ln y} (\ln y^*) &= \left(-\frac{1-\alpha}{\alpha} \right) \alpha s e^{-\frac{1-\alpha}{\alpha} \ln y^*} \\ &= -(1-\alpha) s e^{-\frac{1-\alpha}{\alpha} \ln y^*} \\ &= -(1-\alpha) s (y^*)^{-\frac{1-\alpha}{\alpha}}. \end{aligned}$$

But in the steady state $\phi (\ln y^*) = 0 \Leftrightarrow s (y^*)^{-\frac{1-\alpha}{\alpha}} = (n + \delta + x)$. So

$$s (y^*)^{-\frac{1-\alpha}{\alpha}} = (n + \delta + x),$$

[1] Can't you do nonlinear estimation of this thing? Well, yes. Steve Dowrick goes through this exercise. His note can be downloaded from http://ecocomm.anu.edu.au/economics/staff/dowrick/de-linear.pdf .

hence

$$\phi\prime_{\ln y}\left(\ln y^*\right) = -\left(1-\alpha\right)\left(n+\delta+x\right).$$

Substituting these results back into equation (3) yields.

$$\hat{y} \approx -\left(1-\alpha\right)\left(n+\delta+x\right)\left(\ln y - \ln y^*\right). \tag{4}$$

The rate of of convergence (RoC) tells us how quickly the economy is approaching its steady state. Specifically, its defined as the proportional change in the growth rate of income (per efficiency unit) from a change in income

$$\frac{d\hat{y}}{d\ln y} = -\left(1-\alpha\right)\left(n+\delta+x\right) \equiv \lambda.$$

A few remarks on RoC : The rate of convergence declines (nummerically) when α rises, and vice versa. Why? Suppose α is relatively large, the aggregate production function is then less sharply curbed, and diminishing returns sets in slowly. As a result, the *average* product of capital will change only "little" if we increase the capital stock (in efficiency units), which implies that the induced change in the growth rate, from a change in the capital stock is small: the rate of convergence is therefore low. The rate of convergence also depends on n and δ. A high rate of growth in the labor force, will lower the steady state level of capital per efficiency unit of labor and will therefore increase the average product of capital in the vicinity of the steady state. As a result, near the steady state growth in capital per efficiency unit of labor will be more sensitive to changes in the capital stock, the rate of convergence is higher. A high δ or x leads to relatively quick convergence for exactly the same reason.

But we are not done yet. What we ultimately are looking for is an equation of the form

$$\ln\left(\frac{Y(t)}{L(t)}\right) - \ln\left(\frac{Y(0)}{L(0)}\right) = \beta_0 + \beta_1 \ln\left(\frac{Y(0)}{L(0)}\right) + \beta_2 \ln s + \beta_2\left(n+x+\delta\right). \tag{5}$$

In order to accomplish this task a few more calculations are needed.

Maybe its already obvious that equation (4) is just a first order differential equation with constant coefficient? To be sure, define

$$\begin{aligned}\ln y &= x \\ \hat{y} &= \dot{x} \\ \lambda &\equiv \left(1-\alpha\right)\left(n+\delta+x\right) \\ b &= \left(1-\alpha\right)\left(n+\delta+x\right)\ln y^*,\end{aligned}$$

and we are left with is the following equation:

$$\dot{x}(t) = -\lambda x(t) + b,$$

This we can solve easily

$$x(t) = x(0)\, e^{-\lambda t} + \frac{b}{\lambda}\left(1 - e^{-\lambda t}\right)$$

Reinserting the stuff we got rid of:[2]

$$\ln y(t) = \ln y(0)\, e^{-\lambda t} + \ln y^* \left(1 - e^{-\lambda t}\right)$$

In terms of a growth rate

$$\ln y(t) - \ln y(0) = \left(e^{-\lambda t} - 1\right) \ln y(0) + \left(1 - e^{-\lambda t}\right) \ln y^*.$$

Finally, using our knowledge of the steady state level of income per efficiency unit: $y^* = \left(\frac{s}{n+\delta+x}\right)^{\frac{\alpha}{1-\alpha}}$, it follows that

$$\begin{aligned}
\ln y(t) - \ln y(0) &= \left(e^{-\lambda t} - 1\right) \ln y(0) + \left(1 - e^{-\lambda t}\right) \ln y^* \\
&\Downarrow \\
\text{using that } \ln y(t) - \ln y(0) &= \ln\left(\frac{Y(t)}{L(t)}\right) - \ln\left(\frac{Y(0)}{L(0)}\right) - \ln A(t) + \ln A(0) \\
&\Downarrow \\
\ln\left(\frac{Y(t)}{L(t)}\right) - \ln\left(\frac{Y(0)}{L(0)}\right) &= \left[xt + \ln A(0)\left(1 - e^{-\lambda t}\right)\right] - \left(1 - e^{-\lambda t}\right) \ln \frac{Y(0)}{L(0)} \\
&\quad + \left(1 - e^{-\lambda t}\right) \frac{\alpha}{1-\alpha} \ln\left(\frac{s}{n+\delta+x}\right)
\end{aligned}$$

If we define

$$\begin{aligned}
\left[xt + \ln A(0)\left(1 - e^{-\lambda t}\right)\right] &\equiv \beta_0 \\
-\left(1 - e^{-\lambda t}\right) &\equiv \beta_1 \\
\left(1 - e^{-\lambda t}\right) \frac{\alpha}{1-\alpha} &\equiv \beta_2,
\end{aligned}$$

we have equation (5).

Recovering the structural parameters. Taking this equation to the data leads to the results reported in MRW Table IV p. 426. Specifically, in the intermediate sample (column 2):

$$\hat{\beta}_1 = -0.228,$$

[2] In class we used the capital stock rather than output to make a quantitative assessment of how long economies tend to be outside the steady state: $\ln y(t) = \ln y(0)\, e^{-\lambda t} + \ln y^* \left(1 - e^{-\lambda t}\right) \Leftrightarrow \ln y(t) - \ln y^* = \left(\ln y(0) - \ln y^*\right) e^{-\lambda t}$. The time it takes $(t_{1/2})$ to move half way to the steady state

$$\frac{\ln y\left(t_{1/2}\right) - \ln y^*}{\ln y(0) - \ln y^*} = \frac{1}{2} = e^{-\lambda t}$$

or, taking logs and rearrangeing terms:

$$\frac{\ln(1/2)}{-\lambda} = t_{1/2}.$$

Accordingly, the statements that we made holds here as well.

on this basis we can derive the rate of convergence implied by the regression. Being firm believers in the structural model, and using the fact that $t = 25$ (=85-60):

$$\begin{aligned} -\left(1-e^{-\lambda \cdot 25}\right) &= -0.228 \\ &\Downarrow \\ \lambda &\approx 0.01. \end{aligned}$$

Next, note that

$$\frac{\beta_2}{-\beta_1} = \frac{\alpha}{1-\alpha},$$

which mean that we can recover α from (again, using the intermediate sample results)

$$\frac{\hat{\beta}_2}{\hat{\beta}_1} = \frac{0.644}{0.228} = 2.8 = \frac{\alpha}{1-\alpha},$$

implying that

$$\alpha \approx 0.7.$$

Using the estimate for $\ln(n+\delta+x)$ leads to essentially the same result (0.67).

Three things are worth noting. First, we will not be able to reject the restriction that the coefficient (numerically) on the investment rate equals that of $\ln(n+\delta+x)$. The 95% confidence interval for the estimate related to $\ln s$ is 0.644 ± 0.104 while that for $\ln(n+\delta+x)$ is 0.464 ± 0.307. Clearly they intersect. Second, however, $\alpha = .7$ is a value too high to be consistent with national accounts data, where capital's share is roughly 1/3. This result mirrors the conclusion from the levels-regression (Table I). Third, note that $\lambda = 0.01$ is far too low a rate of convergence to be consistent with priors. Using "standard parameter values" for the parameters entering λ we would expect something like:

$$\lambda = (1-\alpha)(n+x+\delta) = \frac{2}{3}(0.01+0.02+0.05) \approx 0.05.$$

2 The Augmented Solow Model: Guesstimating Human Capitals' Share

In Mankiw et al (1992) we are working with a production function of the following form

$$Y = K^{\alpha} H^{\beta} (AL)^{1-\alpha-\beta} \tag{6}$$

Note that we maintain CRTS in all rival inputs, now including human capital. Since we maintain perfect competition we have that

$$\begin{aligned} Y &= \text{compensation for physical capital} \\ &\quad + \text{compensation for labor input,} \end{aligned}$$

where the latter includes compensation for both human capital and raw labor (brains and brawn). Since

$$\text{compensation for physical capital}=\frac{\frac{dY}{dK}K}{Y}=\alpha$$

it follows that

$$\text{compensation for labor input} = 1-\alpha.$$

At the lectures we've talked about the fact that $\alpha = 1/3$ might be a reasonable rule-of-thumb. The problem is however, that we now have to distribute the remaining 2/3 between compensation for human capital, and compensation for "raw" labor, as

$$\begin{aligned} \text{compensation for labor input} &= \frac{\frac{dY}{dH}H}{Y}+\frac{\frac{dY}{dL}L}{Y} \\ &= \beta+(1-\alpha-\beta)\,. \end{aligned}$$

Mankiw et al make the following back-of-the envelope calculation to establish a prior regarding the value of *human capital's* share:

$$\beta \approx (1-\alpha)\left(1-\frac{\text{minimum wage}}{\text{average wage in manufacturing}}\right),$$

where the minimum wage is to be thought of as a proxy for the wage of an individual (virtually) without any human capital. But where does this approximation come from?

We begin by rewriting the production function somewhat

$$Y = K^{\alpha}\left(eL\right)^{1-\alpha}, \tag{7}$$

where the efficiency of each unit of labor, e, is

$$e=\left(\frac{H}{L}\right)^{\frac{\beta}{1-\alpha}}A^{\frac{1-\alpha-\beta}{1-\alpha}}, \tag{8}$$

i.e. consists of human capital *per person* ($H/L \equiv h$) along with technology A. You can check that equations (8) and (7) together are equivalent to equation (6). Its further convenient, to rewrite (7) to yield[3]

$$\frac{Y}{L}=\left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}}e.$$

[3] The steps in the rewrite are: first divide equation (7) through by L. Next divide and multiply on the right hand side by $(Y/L)^{\alpha}$. Isolating Y/L on the left hand side leads to the stated result.

Now, under perfect competition, the wage is given by

$$\begin{aligned} w &= \frac{dY}{dL} = (1-\alpha)\frac{Y}{L} = (1-\alpha)\left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}} e \\ &= (1-\alpha)\left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}} h^{\frac{\beta}{1-\alpha}} A^{\frac{1-\alpha-\beta}{1-\alpha}}. \end{aligned}$$

Hence, we are implicitly assuming that human capital and labor are supplied simultanously. Accordingly, each unit of labor is accompinied by h units of human capital; higher levels of human capital per person are associated with a higher wage.

Next, imagine that we are comparing two workers at different skill levels, i.e. equipped with different levels of human capital: h^s and h^u (**s**killed and **un**skilled, respectively). Imagine further more, that they work with the same capital and technology. The wage of a worker of type $i = s, u$ is

$$w^i = (1-\alpha)\left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}} \left(h^i\right)^{\frac{\beta}{1-\alpha}} A^{\frac{1-\alpha-\beta}{1-\alpha}}.$$

Take logs and calculate the difference in implied wage

$$\ln w^u - \ln w^s = \frac{\beta}{1-\alpha}\left(\ln h^u - \ln h^s\right)$$

which is the same as

$$\frac{w^u - w^s}{w^s} = \frac{\beta}{1-\alpha}\frac{h^u - h^s}{h^s}$$

or

$$\frac{w^s - w^u}{w^s} = \left(1 - \frac{w^u}{w^s}\right) = \frac{\beta}{1-\alpha}\left(\frac{h^s - h^u}{h^s}\right) = \frac{\beta}{1-\alpha}\left(1 - \frac{h^u}{h^s}\right).$$

Now imagine that the unskilled laborer has very few units of human capital – are virtually, but not entirely, without skills. Then we may assume that

$$\frac{h^u}{h^s} \approx 0.$$

We thus have

$$\left(1 - \frac{w^u}{w^s}\right)(1-\alpha) = \beta.$$

MRW argue that the minimum wage is a sensible approximation for the wage of an unskilled worker, w^u, while the average wage in manufacturing serves as a proxy for the skilled worker. They posit that

$$\frac{w^u}{w^s} \in (0.3, 0.5).$$

If $\alpha = 1/3$ it follows that

$$\beta \in \left((1-0.5)\frac{2}{3}, (1-0.3)\frac{2}{3}\right),$$

or ranges from 1/3 to roughly 1/2.

3 The Rate of Convergence in the Augmented Solow Model

This is going to be slightly more painfull since we have two differential equations. But the methodology is basically the same.

We begin with the production function:

$$y = k^{\alpha} h^{\beta},$$

where $h = H/AL$. As before, note that the growth rate in income per efficiency units of labor is given by

$$\hat{y} = \alpha \hat{k} + \beta \hat{h}. \tag{9}$$

The two fundamental laws of motion are

$$\dot{h} = s_h y - (n + \delta + x) h,$$

$$\dot{k} = s_k y - (n + \delta + x) h.$$

Substituting these back into equation (9):

$$\begin{aligned} \hat{y} &= \alpha \left[s_k \left(\frac{y}{k} \right) - (n + \delta + x) \right] \\ &\quad + \beta \left[s_h \left(\frac{y}{h} \right) - (n + \delta + x) \right] \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} \hat{y} &= \alpha \left[s_k k^{\alpha - 1} h^{\beta} - (n + \delta + x) \right] \\ &\quad + \beta \left[s_h k^{\alpha} h^{\beta - 1} - (n + \delta + x) \right]. \end{aligned}$$

As before, we'll do a log-linearization, for exactly the same reason as above:

$$\begin{aligned} \hat{y} &= \alpha \left[s_k e^{(\alpha - 1) \ln k + \beta \ln h} - (n + \delta + x) \right] \\ &\quad + \beta \left[s_h e^{\alpha \ln k + (\beta - 1) \ln h} - (n + \delta + x) \right] \equiv \theta (\ln k, \ln h) \end{aligned}$$

So approximately

$$\begin{aligned} \hat{y} &\approx \theta (\ln k^*, \ln h^*) + \theta'_{\ln k} (\ln k^*, \ln h^*) (\ln k - \ln k^*) \\ &\quad + \theta'_{\ln h} (\ln k^*, \ln h^*) (\ln h - \ln h^*). \end{aligned} \tag{10}$$

Now we have to calculate $\theta (\ln k^*, \ln h^*)$, $\theta'_{\ln k} (\ln k^*, \ln h^*)$ and $\theta'_{\ln h} (\ln k^*, \ln h^*)$. Here we go.

It should be clear that

$$\theta (\ln k^*, \ln h^*) = 0.$$

(Why?)

Next:

$$\begin{aligned}\theta'_{\ln k}\left(\ln k, \ln h\right) &= \alpha s_k\left(\alpha-1\right)e^{(\alpha-1)\ln k+\beta\ln h} \\ &\quad +\beta\alpha s_h e^{\alpha\ln k+(\beta-1)\ln h}\end{aligned}$$

imposing steady state:

$$\begin{aligned}\theta'_{\ln k}\left(\ln k^*, \ln h^*\right) &= \alpha\left(\alpha-1\right)s_k e^{(\alpha-1)\ln k^*+\beta\ln h^*} \\ &\quad +\beta\alpha s_h e^{\alpha\ln k^*+(\beta-1)\ln h^*},\end{aligned}$$

and noting that $s_k e^{(\alpha-1)\ln k^*+\beta\ln h^*} = s_k\left(\frac{y}{k}\right)^* = (n+\delta+x)\,, s_h e^{\alpha\ln k^*+(\beta-1)\ln h^*} = s_h\left(\frac{y}{h}\right)^* = (n+\delta+x)$, mean that

$$\theta'_{\ln k}\left(\ln k^*, \ln h^*\right) = (\alpha-1)\,\alpha\,(n+\delta+x)+\beta\alpha\,(n+\delta+x) = (\beta+\alpha-1)\,(n+\delta+x)\,\alpha.$$

On for the next one. Same steps:

$$\begin{aligned}\theta'_{\ln h}\left(\ln k, \ln h\right) &= \alpha s_k\beta e^{(\alpha-1)\ln k+\beta\ln h} \\ &\quad +\beta\left(\beta-1\right)s_h e^{\alpha\ln k+(\beta-1)\ln h}\end{aligned}$$

Imposing steady state:

$$\theta'_{\ln h}\left(\ln k^*, \ln h^*\right) = \alpha\beta s_k e^{(\alpha-1)\ln k^*+\beta\ln h^*} + \beta\left(\beta-1\right)s_h e^{\alpha\ln k^*+(\beta-1)\ln h^*}$$

Since $s_k e^{(\alpha-1)\ln k^*+\beta\ln h^*} = (n+\delta+x)\,, s_h e^{\alpha\ln k^*+(\beta-1)\ln h^*} = (n+\delta+x)$, so

$$\begin{aligned}\theta'_{\ln h}\left(\ln k^*, \ln h^*\right) &= \beta\alpha\,(n+\delta+x)+\beta\,(\beta-1)\,(n+\delta+x) \\ &= \beta\,(\alpha+\beta-1)\,(n+\delta+x)\end{aligned}$$

Inserting $\theta\left(\ln k^*, \ln h^*\right)$, $\theta'_{\ln k}\left(\ln k^*, \ln h^*\right)$ and $\theta'_{\ln h}\left(\ln k^*, \ln h^*\right)$ into equation (10):

$$\begin{aligned}\hat{y} &\approx 0+(\beta+\alpha-1)\,(n+\delta+x)\,\alpha\,(\ln k-\ln k^*) \\ &\quad +\beta\,(\alpha+\beta-1)\,(n+\delta+x)\,(\ln h-\ln h^*)\,.\end{aligned}$$

Collecting terms:

$$\hat{y} \approx (\beta+\alpha-1)\,(n+\delta+x)\left[\alpha\,(\ln k-\ln k^*)+\beta\,(\ln h-\ln h^*)\right].$$

From the production function $y = h^\beta k^\alpha$ it follows that

$$\ln y-\ln y^* = \alpha\,(\ln k-\ln k^*)+\beta\,(\ln h-\ln h^*)$$

so

$$\hat{y} \approx (\beta+\alpha-1)\,(n+\delta+x)\left[\ln y-\ln y^*\right]. \tag{11}$$

Consequently, the rate of convergence is

$$\frac{\partial \hat{y}}{\partial \ln y} = -\left(1-\beta-\alpha\right)\left(n+\delta+x\right) \equiv \lambda^{MRW}.$$

Exercise: derive equation (16) in Mankiw, Romer and Weil (1992). *Hint:* start by solving the differential equation (11); this gives you equation (14) in their paper. Next, use what you know about y^*, to end up with their equation 16.